THERMOSOLUTAL CONVECTION IN RIVLIN-ERICKSEN VISCO- ELASTIC FLUID IN POROUS MEDIUM

A DISSERTATION

SUBMITTED TO THE HIMACHAL PRADESH UNIVERSITY, SHIMLA
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

MASTER OF PHILOSOPHY

IN

MATHEMATICS

OCTOBER, 2002

Supervised By :
Dr. R.C. Sharma
(Professor & Chairman)

Submitted By :
Poonam Sharma
(M.Sc.)

DEPARTMENT OF MATHEMATICS
HIMACHAL PRADESH UNIVERSITY
SUMMER HILL, SHIMLA-171005

Dedicated

To

My Loving Parents

CERTIFICATE

This is to certify that the work undertaken in the dissertation entitled **"THERMOSOLUTAL CONVECTION IN RIVLIN-ERICKSEN VISCO-ELASTIC FLUID IN POROUS MEDIUM"**, by Poonam Sharma, has been carried under my supervision and has not been submitted elsewhere for a degree or diploma.

Date : 9. 10. 2002
Place : Shimla

(Dr. R.C. SHARMA)
Professor and Chairman
Department of Mathematics,
Himachal Pradesh University,
Summer Hill, Shimla-171005

ACKNOWLEDGEMENTS

I express my sincere gratitude to **Dr. R. C. Sharma**, Professor and Chairman, **Department of Mathematics, Himachal Pradesh University, Shimla-171005** under whose supervision the present work has been carried out. His polite behaviour, helpful criticism and constant encouragement helped me immensely to accomplish this work.

I express my deep gratitude to my **parents** and all other **family members** including **Vikram Bhai** for their blessings and timely help to meet the basic requirements.

I am also highly indebted to my colleagues of Department of Mathematics for their valuable and concrete suggestions in preparing this present work.

I will be failing in my duty if I don't thank my sister **Anju** and Friends **Shruti and Neelam** for their untiring efforts in completing this work.

Finally, I would like to thank Mr. Arvind for putting these written words into beautiful printing work in shortest possible time, inspite of his busy schedule.

Date:

Place: Shimla

(Poonam Sharma)

CONTENTS

Chapter No	*Title of the Chapter*	*Page No*
1	**Introduction**	**1-17**
2.	**Thermosolutal Convection in Rivlin-Ericksen fluid in porous Medium in Hydromagnetics**	**18-31**
3.	**Thermosolutal Instability of Rivlin-Ericksen rotating fluid in Porous medium**	**32-43**
	References	**44-45**

CHAPTER - I

INTRODUCTION

STABILITY OF HYDRODYNAMICAL SYSTEM

As every system in nature is subject to many small perturbations, the investigation of stability of a physical system is of great importance. For an equilibrium state or a steady flow to be of permanent type, it must not only satisfy the mechanical equations but must also be stable against the arbitrary small perturbations. A large body of theoretical work by Lin (1955), Chandrasekhar (1981), Joseph (1976), Drazin and Reid (1981), Gershuni and Zhukhovitskii (1976) has been developed in an attempt to understand and predict phenomena of stability or instability.

Consider a hydrodynamic system in a stationary state i.e. one in which the variables defining the configuration are not function of the time. To investigate its stability, we must determine the reactions of the system to arbitrary small perturbations. If the perturbations gradually die down or if the system never departs appreciably from this stationary state, the system is said to be stable. If the perturbations grow with time or if the system departs more and more from initial state, we may say that the system is unstable. If the system neither tends to return to its initial position nor departs from its disturbed position, the system is said to be in neutral equilibrium. Clearly, a system is unstable even if there is only one special mode of disturbance with respect to which it is unstable and a system is stable unless, it is stable with respect to every possible disturbance to which it can be subject. The locus which separates the stable and unstable state defines the state of marginal stability or neutral stability.

The equations of motion and boundary conditions are usually expressed in non-dimensional form involving a number of dimensionless parameters R_1, R_2,.....e.g. the Reynolds number, the Rayleigh number, the Taylor number etc. The stability of the system can then be examined in various stages by treating all the parameters except one, say R_i, as constant. If there exists a value R_i^c of this parameter which separates state of stable equilibrium from those of unstable equilibrium the state corresponding to the critical value R_i^c is called a state of neutral or marginal stability.

For the mathematical treatment of a problem, generally we start from an initial flow which represents a stationary state of the system. By supposing that various physical variables describing the flow suffer infinitesimal increments, we first obtain the equations governing these increments. In obtaining these equations from the relevant equations of motion and boundary conditions, we neglect all products, square and higher powers of the increments and retain only those terms which are linear in them. The equations so obtained are called linearized perturbation equations and the theory derived on the basis of such linearized equations is called linear stability theory.

If perturbation quantities vary as

$$e^{ik_x x + ik_y y + nt}$$

where k= $\left(k_x^2 + k_y^2\right)^{1/2}$ is the wave number of perturbation and n is, in general, a complex constant. It is related to wavelength λ by the relation

$$\lambda = 2\pi / k$$

The relation between n and k is called dispersion relation. The parameter n is function of k (wave number) and other parameters defining the system. If the value determined for n by the dispersion relation is:

i) Real and negative, the system is stable.

ii) Real and positive, the system is unstable.

iii) complex, say

$n = n_r + in_i$, where n_r and n_i are real and

(a) $n_r<0$, the system is stable.

(b) $n_r>0$, the system is unstable.

(c) $n_r = 0$, $n_i \neq 0$, the modes are oscillatory.

If in a given problem it can be established that n is real, then $n = 0$ will separate the stable and unstable modes and we can study the marginal state by putting $n = 0$. This is called "principle of exchange of stabilities".

If $n = in_i$ and at the onset of instability, the modes are oscillatory with increasing amplitude, we have the case of overstability.

THERMAL INSTABILITY

The problem of the onset of thermal instability in horizontal layers of fluid heated from below is well suited to illustrate many facts, mathematical and physical, of the general theory of hydrodynamic stability.

Consider a horizontal layer of fluid in which an adverse temperature gradient is maintained by heating it from below. The temperature gradient thus maintained is qualified as adverse since, on account of thermal expansion, the fluid at the bottom will be lighter than the fluid at the top and this is a top-heavy arrangement which is potentially unstable. Because of this instability, there will be a tendency on the part of the fluid to redistribute itself and remedy the weakness in its arrangement. However, this tendency on the part of the fluid will be inhibited by its own viscosity. Hence, a certain critical temperature gradient must be exceeded before the instability can manifest itself.

Bénard (1900) performed the experiments on the onset of thermal instability in fluids and established two facts:

i) a certain critical adverse temperature gradient must be exceeded before instability can set in.

ii) The motions, which ensue on surpassing the critical temperature gradient, have a stationary cellular character.

Lord Rayleigh (1916) gave the theoretical foundations for a correct interpretation of above facts and showed that, what decides the stability or otherwise, of a layer of fluid heated from below is the numerical value of the non-dimensional parameter

$$R = \frac{g\alpha\beta d^4}{\nu\kappa} \tag{1}$$

Here R is called the Rayleigh number, β (= | dT/dz |) is the uniform temperature gradient which is maintained, g is the acceleration due to gravity, d is the depth of the layer and α, κ, ν are the coefficients of volume expansion, thermal diffusivity and kinematic viscosity respectively. Rayleigh showed that instability must set in when R exceeds a certain critical value R_c ; and that when R just exceeds R_c, a stationary pattern of motion must come to prevail. This is the Bénard convection or thermal instability problem.

BASIC EQUATIONS OF HYDRODYNAMICS FOR THERMAL INSTABILTY

For a general treatment of the problem of thermal instability, we need the basic equations of viscous fluid dynamics. Let $p, \rho, T, \vec{v}\,(u,v,w)$ denote respectively the fluid pressure, fluid density, fluid temperature and velocity of the fluid.

i) EQUATION OF CONTINUITY

The equation of continuity representing the conservation of mass is

$$\frac{\partial \rho}{\partial t} + (\vec{v}.\nabla)\rho + \rho\nabla.\vec{v} = 0. \tag{2}$$

For incompressible fluid, the equation of continuity reduces to

$$\nabla.\vec{v} = 0. \tag{3}$$

ii) EQUATION OF MOTION

The equations of motion are

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right] = -\text{grad } p + \rho\,\vec{g} + \frac{1}{3}\mu \text{ grad div } \vec{v} + \mu\nabla^2\vec{v}, \tag{4}$$

where $\vec{g}$ is the acceleration due to gravity which is external force acting on the fluid.

For incompressible fluid, equation (4) reduces to

$$\rho\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}.\nabla)\vec{v}\right]=-\text{grad } p+\rho\,\vec{g}+\mu\nabla^2\vec{v}. \tag{5}$$

These equations are called Navier-Stokes equations and are three in number. Here μ is the coefficient of viscosity.

iii) EQUATION OF HEAT CONDUCTION

The law of conservation of energy leads to the equation of heat conduction

$$\rho\left[\frac{\partial}{\partial t}+\vec{v}.\nabla\right](C_v T)=-p\text{ div }\vec{v}+\nabla.(K\nabla T)+\phi, \tag{6}$$

where C_v is the specific heat at constant volume, T is the temperature, K is the coefficient of thermal conductivity and ϕ is the viscous dissipation function given by

$$\phi = 2\mu e_{ij}^2-\frac{2}{3}\mu e_{jj}^2, \tag{7}$$

where e_{ij} = rate of strain tensor $=\frac{1}{2}\left[\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right]$

For incompressible fluid, equation (6) reduces to

$$\left[\frac{\partial}{\partial t}+\vec{v}.\nabla\right]T=\kappa\nabla^2 T+\frac{1}{\rho C_v}\phi \tag{8}$$

$\kappa=\dfrac{K}{\rho C_v}$ is the thermal diffusivity and $\phi = 2\mu\,(e_{ij})^2$.

iv) EQUATION OF STATE

The equation of state, for substances with which we shall be principally concerned, is

$$\rho = \rho_o[1-\alpha(T-T_o)], \tag{9}$$

where T_o is the temperature at which $\rho=\rho_o$ and α is the coefficient of volume expansion.

BOUSSINESQ APPROXIMATION

The density may be treated as a constant in all terms in the equations of motion except in the external force. This is the Boussinesq approximation. The equations of viscous, compressible fluid dynamics are quite complicated but if we apply the Boussinesq approximation, the equations become simplified considerably.

The situations (when equations get simplified considerably) occur when the variability in the density and in the various coefficients is due to variations in the temperature of only moderate amounts say 10°. The origin of the simplifications in these cases is due to the smallness of the coefficient of volume expansion α. For gases and liquids such as we shall be mostly concerned with α is in the range 10^{-3} to 10^{-4}. If the temperature variations do not exceed 10°, the variations in density are at most one percent. The variations of this small amount can be ignored. But the density changes in the external force term $\rho\vec{g}$ in the equation of motion cannot be ignored as $|\vec{g}| \sim 10^3 \text{cm/sec}^2$, this is because the acceleration resulting from $(\delta\rho)X_i = \alpha\Delta T X_i$ (where ΔT is a measure of the variations in temperature which takes place)can be quite large, larger than, for example, the acceleration due to the inertial term $u_j\ \partial u_i/\partial x_j$ in the equation of motion. Under the Boussinesq approximation, the basic hydrodynamical equations are:

EQUATION OF CONTINUITY

$$\frac{\partial v_j}{\partial x_j} = 0, \tag{10}$$

EQUATION OF MOTION

$$\frac{\partial v_i}{\partial t} + vj\frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x_i} + \left(1 + \frac{\delta\rho}{\rho_o}\right)X_i + v\nabla^2 v_i, \tag{11}$$

where ρ_o is the density at some properly chosen mean temperature T_o and

$$\delta\rho = -\rho_o\,\alpha\,(T - T_o), \tag{12}$$

and $v = \mu/\rho_o$ denotes the kinematic viscosity.

EQUATION OF HEAT CONDUCTION

$$\frac{\partial T}{\partial t} + v_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T, \tag{13}$$

where $\kappa = K/\rho_o C_v$ is the coefficient of thermometric conductivity.

PERTURBATION EQUATIONS

Consider an infinite horizontal layer of fluid in which a steady adverse temperature gradient β is maintained, further, let there be no motions. The initial state is given by

$$v_j = 0 \text{ and } T = T(\lambda_j x_j), \tag{14}$$

where $\lambda = (0,0,1)$ is a unit vector in the direction of the vertical. We have the following set of hydrodynamical equations in the initial state.

$$\left.\begin{aligned} & u = v = w = 0 \\ & p = p_o - g\rho_o [\lambda_i x_i + \frac{1}{2} \lambda_i \lambda_j x_i x_j] \\ & T = T_o - \beta \lambda_j x_j \\ & \rho = \rho_o [1 + \alpha\beta\lambda_j x_j]. \end{aligned}\right\} \tag{15}$$

After perturbating slightly the initial state, we have the following linearized form of perturbed hyrodynamical equations.

$$\left.\begin{aligned} & \frac{\partial v_j}{\partial x_j} = 0, \\ & \frac{\partial v_i}{\partial t} = -\frac{\partial}{\partial x_i}\left[\frac{\delta p}{\rho_o}\right] + g\alpha\theta\lambda + v\nabla^2 v_i, \\ & \frac{\partial \theta}{\partial t} = \beta w + \kappa \nabla^2 \theta. \end{aligned}\right\} \tag{16}$$

where δp, $\delta \rho$ and θ are perturbations in the pressure distribution, density and temperature and $w(=\lambda_j v_j)$ is the z-component of the velocity.

THE BOUNDARY CONDITIONS

The fluid is confined between the planes $z = 0$ and $z = d$. Regardless of the nature of these bounding surface, we must require

$$\theta = 0 \text{ and } w = 0 \text{ for } z = 0 \text{ and } z = d, \tag{17}$$

because the surface $z = 0$ and d are maintained at constant temperatures so they can not suffer any change in temperature and the normal component of the velocity must vanish on these surfaces. We will now distinguish two kinds of bounding surfaces.

Consider first a rigid surface on which no slip occurs and this implies that not only w, but also the horizontal components of the velocity u and v vanish

So the equation of continuity gives

$$\frac{\partial w}{\partial z} = 0, \text{ on a rigid surface.} \tag{18}$$

Also it follows that

$$\zeta = 0, \text{ on a rigid surface.} \tag{19}$$

Secondly, consider the free surface on which no tangential stresses act.

i.e. $\tau_{xz} = \tau_{yz} = 0.$

Since w vanishes for all x and y on the bounding surface, it follows that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \text{ on a free surface} \tag{20}$$

So the equation of continuity on differentiating w.r.t. z and using (20) gives

$$\frac{\partial^2 w}{\partial z^2} = 0, \text{ on a free surface.} \tag{21}$$

Also ζ (z-component of vorticity) = $\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

$$\therefore \frac{\partial \zeta}{\partial z} = 0, \quad \text{on a free surface.} \tag{22}$$

The perturbation equations (16) can be written as

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_o}\frac{\partial}{\partial x}\delta p + v\nabla^2 u, \tag{23}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_o}\frac{\partial}{\partial y}\delta p + v\nabla^2 v, \tag{24}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_o}\frac{\partial}{\partial z}\delta p + v\nabla^2 w + g\alpha\theta, \tag{25}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{26}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa\nabla^2\theta. \tag{27}$$

Operating equations (23) and (24) by $-\partial/\partial x$ and $-\partial/\partial y$ respectively, adding and using (26), we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial w}{\partial z}\right] = \frac{1}{\rho_o}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\delta p + v\nabla^2\left(\frac{\partial w}{\partial z}\right) \tag{28}$$

Eliminating δp between (25) and (28), we get

$$\frac{\partial}{\partial t}\left(\nabla^2 w\right) = g\alpha\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right] + v\nabla^4 w. \tag{29}$$

Operating (23) by $-\partial/\partial y$ and (24) by $\partial/\partial x$ and adding, we obtain

$$\frac{\partial \zeta}{\partial t} = v\nabla^2\zeta, \tag{30}$$

where $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ is the z-component of vorticity.

Equations (27), (29) and (30) are the required perturbation equations whose solutions must satisfy must satisfy the boundary conditions mentioned above.

NORAML MODE ANALYSIS

We analyse an arbitrary disturbance into a complete set of normal modes and examine the stability of each of these modes, individually. The analysis can be made in terms of two-dimensional periodic waves of assigned wave numbers. Thus we ascribe to all quantities describing the perturbation a dependence on x,y and t of the form

$$\exp[ik_x x + ik_y y + nt], \tag{31}$$

where

$$k = \sqrt{k_x^2 + k_y^2}$$

is the wave number of the disturbance and n is a constant. We suppose that the perturbations θ,w and ζ have the forms

$$[w, \theta, \zeta] = [W(z), \Theta(z), Z(z)] \exp[ik_x x + ik_y y + nt]. \tag{32}$$

For functions with this dependence on x,y and t

$$\frac{\partial}{\partial t} = n, \ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2 \text{ and } \nabla^2 = \frac{d^2}{dz^2} - k^2, \tag{33}$$

equations (27),(29) and (30) become

$$n\left(\frac{d^2}{dz^2} - k^2\right)W = -g\alpha k^2\Theta + v\left(\frac{d^2}{dz^2} - k^2\right)^2 W, \tag{34}$$

$$n\Theta = \beta W + \kappa\left(\frac{d^2}{dz^2} - k^2\right)\Theta, \tag{35}$$

$$nZ = v\left(\frac{d^2}{dz^2} - k^2\right)Z. \tag{36}$$

The boundary conditions become

$$\Theta = 0 \text{ and } W = 0, \text{ for } z = 0 \text{ and } z = d. \tag{37}$$

and

$$\left.\begin{aligned} &Z = 0 \text{ and } \frac{dW}{dz} = 0, \text{ on a rigid surface} \\ &\frac{dZ}{dz} = 0 \text{ and } \frac{d^2W}{dz^2} = 0, \text{ on a free surface} \end{aligned}\right\} \tag{38}$$

We now transform equations (34)-(36) and boundary conditions (37)-(38) in non-dimensional variables. Measuring length in the unit d i.e.

$x = x'd$, $y = y'd$, $z = z'd$ and let

$$a = kd \text{ and } \sigma = nd^2/\nu, \tag{39}$$

where a and σ are the wave number and time constant. Equations (34) and (35) become

$$(D^2 - a^2)(D^2 - a^2 - \sigma)W = \left(\frac{g\alpha d^2}{\nu}\right)a^2\Theta, \tag{40}$$

$$(D^2 - a^2 - p_1\sigma)\Theta = -\left(\frac{\beta d^2}{\kappa}\right)W, \tag{41}$$

where $D = d/dz'$ and $p_1 = \nu/\kappa$ is the Prandtl number.

The associated boundary conditions are

$$\Theta = 0, \; W = 0 \text{ for } z = 0 \text{ and } z = 1 \tag{42}$$

and

$$DW = 0 \text{ for } z = 0 \text{ and } z = 1 \text{ if both bounding surfaces are rigid.} \tag{43}$$

or

$$DW = 0 \text{ for } z = 0 \text{ and } D^2W = 0 \text{ for } z = 1 \tag{44}$$

if the bottom surface is rigid and the top surface is free. By eliminating Θ between equations (40) and (41), we obtain

$$(D^2 - a^2)(D^2 - a^2 - \sigma)(D^2 - a^2 - p_1\sigma)W = -Ra^2W, \tag{45}$$

where

$$R = \frac{g\alpha\beta}{\kappa\nu} d^4 \quad \text{is the Rayleigh number.}$$

HYDROMAGNETICS

Hydromagnetics is the branch of science which deals with the motion of electrically conducting fluid in the presence of magnetic field.

Hydromagnetics is the union of two fields of science, namely electromagnetic theory and hydrodynamics. The systematic study of hydromagnetics started only after 1942 when Alfvén combined the two subjects by considering the motion of highly electrically conducting fluid in the presence of a magnetic field. At the beginning, these two branches were being developed independently of each other. Thus it inherits the richness as well as the difficulties of the parent subjects. It is a well-known result in electromagnetic theory that when a conductor moves in a magnetic filed, electric currents are induced in it. These currents experience a mechanical force, called the Lorentz force, due to the presence of the magnetic field. This force tends to modify the initial motion of a conductor. Moreover, the induced currents generate their own magnetic field, which is added on to the primitive magnetic field. It is this coupling between the electromagnetic and mechanical forces which characterizes hydromagnetic phenomena.

Equations of motion are

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right] = -\nabla p + \rho\vec{X} + \mu\nabla^2\vec{v} + \mu \text{ grad } (\text{div } \vec{v}) - \frac{2}{3}\mu \text{ grad } (\text{div } \vec{v})\delta_{ij} + \frac{\mu_e}{4\pi}(\nabla \times \vec{H}) \times \vec{H}, \tag{46}$$

where the last term in the equation represents Lorentz force $\vec{J} \times \vec{B}$ which on using $4\pi\vec{J} = \text{Curl}\,\vec{H}$ and $\vec{B} = \mu_e\vec{H}$ becomes $(\mu_e/4\pi)(\nabla \times \vec{H}) \times \vec{H}$.

Maxwell's equations are

$$\left.\begin{aligned} &\frac{\partial \vec{H}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{H}\right) + \eta \nabla^2 \vec{H}, \\ &\nabla . \vec{H} = 0, \end{aligned}\right\} \tag{47}$$

where η is electrical resistivity and $\vec{v}$ denotes fluid velocity.

NEWTONIAN FLUIDS

Newtonian fluids are those fluids in which there is linear relationship between stress and rate-of-strain. In other words, the stress components are linear functions of the rate-of-strain components. The mathematical formulations of the physical assumptions that are taken to characterize a medium are the constitutive equations. The constitutive equation for an isotropic, Newtonian fluid is

$$\tau_{ij} = 2\mu e_{ij} - \left(\frac{2}{3}\right)\mu e_{kk}\delta_{ij}. \tag{48}$$

This is the required relationship between viscous stress tensor and rate-of-strain tensor.

VISCOELASTIC FLUID

Viscoelastic fluids are those whose behaviour at sufficiently small variable shear stresses can be characterized by three constants(i.e. a coefficient of viscosity, a relaxation time and a retardation time) and whose invariant differential equation of state for general motion are linear in the stresses and includes terms of no higher degree than the second in the stresses and velocity gradients taken together.

The Oldroyd constitutive equations (linearized) are given by

$$\left.\begin{aligned} &T_{ij} = -p\delta_{ij} + \tau_{ij}, \\ &\left(1 + \lambda \frac{\partial}{\partial t}\right)\tau_{ij} = 2\mu\left(1 + \lambda_o \frac{\partial}{\partial t}\right)e_{ij}, \\ &e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \end{aligned}\right\} \tag{49}$$

where T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, μ, λ and λ_o denote respectively the stress tensor, shear stress tensor, rate-of-strain tensor, Kronecker delta, coefficient of viscosity, stress relaxation time and strain retardation time.

POROUS MEDIUM

In order to study the flow of fluids through porous media, it is first of all necessary to clarify that what is understood by the term 'porous medium'. 'Porous Medium' may be defined as solid bodies that contains 'pores' and 'pores' are void spaces which must be distributed more or less frequently throughout the material. Extremely small voids in a solid are called 'molecular interstices' and very large ones are called 'caverns'. 'Pores' are void spaces intermediate in size between caverns and molecular interstices.

The pores in a porous system may be interconnected or non-interconnected. Flow of interstitial fluid is possible only if at least part of the pore space is interconnected. The interconnected part of the pore system is called the effective pore space of the porous medium. The examples of porous media: towers packed with pebbles, Berl saddles, Raschig rings, etc; beds formed of sand, lead shot, etc; porous rocks such as limestone, pumice, dolomite; fibrous aggregates such as cloth, felt, filter paper and finally catalytic particles containing extremely fine 'micro'-pores. The porous medium can be arranged into several classes according to the types of pore spaces which they contain. In addition, pore spaces have also been classified according to whether they are ordered or disordered.

GEOMETRICAL QUANTITIES CHARACTERIZING POROUS MEDIA

i) POROSITY: It is the ratio of the voids to the total volume. It is denoted by $\in$. It is expressed either as a fraction of 1 or in percent. The porosity lies between 0 and 1. In solid physics, one often uses, instead of the porosity, the void ratio e defined by

$$e = \frac{\epsilon}{1-\epsilon}.$$

ii) PORE DIAMETER: The measure of the size of the pores is called pore diameter. This definition is applicable only when all the pores are spherical in shape unless some further specifications are made. If one has the flow of fluids through those pores in mind, however, it will not do to restrict oneself to spherical pores only; the pores must be visualized instead as rather tube-shaped things. One would than call the diameter of such a tube the 'pore-diameter'. It is denoted by δ.

iii) TORTUOSITY: Originally, this was introduced as a kinematical property equal to the relative average length of the flow path of a fluid particle from one side of the porous medium to the other. Thus, it is a dimensionless quantity. It is usually denoted by T.

DIFFERENTIAL FORM OF DARCY'S LAW-ISOTROPIC POROUS MEDIA

Darcy's law, when the separation of the general constant into 'permeability' and 'viscosity' is taken into account, is expressible as follows:

$$q = \frac{Q}{A} = -(k_1/\mu)(p_2 - p_1 + \rho g h)/h. \tag{50}$$

In this form, it applies to horizontal bed of finite thickness h, being percolated by an incompressible liquid of density ρ. This form of law has only a restricted use.

Naturally q will become a vector $\vec{q}$, which is called local filter velocity or seepage velocity. When h becomes infinitesimal, then equation (50) reduces to

$$\vec{q} = -(k_1/\mu)(\text{grad } p - \rho\vec{g}) \tag{51}$$

where $\vec{g}$ is a vector in the direction of gravity (i.e. downward) and of the magnitude of gravity. However, Darcy's experiment does not tell us what happens if the permeability and viscosity are variables. Thus, the coefficient might equally have to be taken into the gradient:

$$\vec{q} = -\text{grad}(k_1 p/\mu) + k_1 \rho g/\mu \tag{52}$$

or

$$\vec{q} = -(k_1 p/\mu)\text{grad}\,\phi \tag{53}$$

where $\phi = gz + \int_{p_0}^{p} dp/\rho(p)$.

is the force potential in which z denotes the vertical coordinate. Consequently, equation (52) can be written in equivalent form as

$$\vec{q} = -\text{grad}\,\psi \tag{54}$$

where $\psi = k_1 p/\mu + \int_{z_0}^{z} k_1 \rho g dz/\mu$,

is the velocity potential.

Both representations by potentials are valid only if the integrals are univalent.

THE BASIC HYDRODYNAMIC EQUATIONS IN POROUS MEDIA

The theory of porous flow is largely based on a generalization of the empirical observations of Darcy (1856). Darcy's law for steady slow flow in homogeneous isotropic material is given by

$$-\rho\vec{g} + \nabla p = -\frac{\mu}{k_1}\vec{q} \tag{55}$$

comparing with the Navier-Stokes equations linearized for slow flow

$$-\rho\vec{g} + \nabla p = \mu\nabla^2\vec{U} \tag{56}$$

It is seen that the Darcy's law replaces the term

$$\mu\nabla^2\vec{U} = \text{div (viscous part of the stress tensor)} \tag{57}$$

with the term

$$-\frac{\mu}{k_1}\vec{q}$$

This term is sometimes regarded as giving the force which the porous solid exerts on the fluid. Equation (57) is unaffected by the addition to $\vec{U}$ of a rigid-body velocity field. It depends only on relative velocities. Analogously, we interpret q to be a velocity measured relative to axes fixed in the porous solid. In the same way that (55) may be said to represent (56) for slow steady flow, it is useful to assume that

$$\rho \frac{d\vec{U}}{dt} = -\nabla p + \rho \vec{g} - \frac{\mu}{k_1}\vec{q} \tag{58}$$

The pore average velocity $\vec{U}$ and the seepage velocity $\vec{q}$ are assumed to be defined at each point and are related by the relation

$$\vec{q} = \phi\, \vec{U} \tag{59}$$

EQUATION OF CONTINUITY

The equation of continuity in porous medium (for compressible fluid) is given by

$$\in \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0. \tag{60}$$

For incompressible fluid, it is given by

$$\nabla . \vec{q} = 0 \tag{61}$$

EQUATION OF HEAT CONDUCTION

The equation of heat conduction is given by

$$[\rho_o C_o \in + \rho_s C_s (1 - \in)]\frac{\partial T}{\partial t} + \rho_o C_o (\vec{q}.\nabla)T = K\nabla^2 T, \tag{62}$$

or

$$E\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \kappa \nabla^2 T, \tag{63}$$

where

$$E = \in + (1 - \in)\frac{\rho_s C_s}{\rho_o C_o} \quad \text{and} \quad \kappa = \frac{K}{\rho_o C_o}$$

Here κ denotes the thermal diffusivity, K is the thermal conductivity and $\rho_o, C_o; \rho_s, C_s$ are density and specific heat at constant pressure for fluid and solid matrix respectively.

CHAPTER - II

Thermosolutal Convection in Rivlin-Ericksen fluid in porous Medium in Hydromagnetics

INTRODUCTION

The thermal convection in a fluid layer in the presence of magnetic field has been discussed exhaustively by Chandrasekhar (1961). Bhatia and Steiner (1973) have studied the thermal instability of a Maxwellian viscoelastic fluid in presence of magnetic field while the thermal convection in Oldroydian viscoelastic fluid in hydromagnetic has been considered by Sharma (1975)

The problem of thermosolutal convection is of great importance because of its application to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layer of the solar atmosphere. The thermosolutal convection problems also arise in oceanography, limnology and engineering.

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigation of such fluids is desirable. The Rivlin-Ericksen fluid is one such fluid. Sharma and Kumar (1996) have studied the thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid subject to a uniform rotation.

In all the above studies, the medium has been considered to be non-porous. When the fluid permeates a porous material, the gross affect is represented by Darcy's law. The problem of thermosolutal convection in fluids in a porous medium is of importance in geophysics, soil sciences, ground water hydrology and astrophysics.

Keeping in mind the importance of non-Newtonian fluid in modern technology and industries, and various applications mentioned above, the thermosolutal instability of a Rivlin-Ericksen fluid in porous medium in the presence of uniform vertical magnetic field has been considered in the present chapter.

PERTURBATION EQUATIONS

We consider an infinite, horizontal incompressible Rivlin-Ericksen fluid layer of thickness d, heated and soluted from below so that, the temperatures, densities and solute concentrations at the bottom surface z = 0 are T_0, ρ_0, C_0 and at the upper surface z = d are T_d, ρ_d and C_d respectively, and that a uniform temperature gradient β $\left(=|dT/dz|\right)$ and a uniform solute gradient β' $\left(=|dC/dz|\right)$ are maintained. The gravity field $\vec{g}$ (0, 0,−g) and uniform vertical magnetic field $\vec{H}$ (0, 0, H) pervade the system. This fluid layer is assumed to be flowing through an isotropic, homogeneous porous medium of porosity $\in$ and medium permeability k_1.

The relevant hydromagnetic equations in porous medium, under Boussinesq approximation, are

$$\frac{1}{\in}\left(\frac{\partial\vec{q}}{\partial t}+\frac{1}{\in}(\vec{q}\cdot\nabla)\vec{q}\right)=-\frac{\nabla p}{\rho_0}+\left(1+\frac{\delta\rho}{\rho_0}\right)\vec{g}-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\vec{q}+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{H}\right)\times\vec{H}, \quad (1)$$

$$\nabla.\vec{q}=0, \quad (2)$$

$$E\frac{\partial T}{\partial t}+\left(\vec{q}.\nabla\right)T=\kappa\nabla^2 T, \quad (3)$$

$$E'\frac{\partial C}{\partial t}+\left(\vec{q}.\nabla\right)C=\kappa'\nabla^2 C, \quad (4)$$

$$\nabla.\vec{H}=0, \quad (5)$$

$$\in\frac{\partial\vec{H}}{\partial t}=\left(\vec{H}.\nabla\right)\vec{q}+\in\eta\nabla^2\vec{H}. \quad (6)$$

where $\nu\left(=\mu/\rho_0\right)$, $\nu'\left(=\mu'/\rho_0\right)$, κ, μ_e, η, $\in$, k_1, κ' denote respectively the kinematic viscosity, kinematic viscoelasticity, thermal diffusivity, magnetic permeability, electrical resistivity, medium porosity, medium permeability and solute diffusivity.

$E = \in + (1-\in)\rho_s c_s / \rho_o c_i$, $\vec{q} = \in \vec{U}$ where ρ_s, c_s and ρ_o c_i stand for density and specific heat for solid and fluid respectively. $\vec{q}$ is the seepage velocity and $\vec{U}$ is the pore average velocity. E' is a constant analogous to E but corresponding to solute rather than heat.

The equation of state is

$$\rho = \rho_o [1 - \alpha (T - T_o) + \alpha' (C - C_o)] \tag{7}$$

where α is the coefficient of volume expansion and T_o, C_o are temperature and solute concentration at which $\rho = \rho_o$. Here the suffix zero refers to values at the reference level $z = 0$.

Initially, $\vec{q} = (0, 0, 0)$, $\rho = \rho(z)$, $p = p(z)$, $T = T(z)$.

Therefore, equations (1) – (6) becomes

$$\left.\begin{aligned}
0 &= -\frac{\nabla p}{\rho_o} + \vec{g} + \frac{\mu_e}{4\pi\rho_0}(\nabla \times \vec{H}) \times \vec{H}, \\
0 &= 0, \\
E\frac{\partial T}{\partial t} &= \kappa \nabla^2 T, \\
E'\frac{\partial C}{\partial t} &= \kappa' \nabla^2 C, \\
\nabla.\vec{H} &= 0, \\
\frac{\partial \vec{H}}{\partial t} &= \eta \nabla^2 \vec{H}.
\end{aligned}\right\} \tag{8}$$

Let $\vec{q}(u, v, w)$, $\delta\rho$, δp, θ, γ and $\vec{h}(h_x, h_y, h_z)$ denote the perturbation in filter velocity $\vec{q}(0,0,0)$, density ρ, pressure p, temperature T, solute concentration C and magnetic field $\vec{H}(0, 0, H)$ respectively.

Substituting the perturbed quantities in equations (1) - (6), we have

$$\frac{1}{\in}\left(\frac{\partial \vec{q}}{\partial t}+\frac{1}{\in}(\vec{q}\cdot\nabla)\vec{q}\right)=-\frac{\nabla(p+\delta p)}{\rho_o}+\left(1+\frac{\delta\rho}{\rho_o}\right)\vec{g}-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\vec{q}+\frac{\mu_e}{4\pi\rho_0}\left[\nabla\times\left(\vec{H}+\vec{h}\right)\right]\times\left(\vec{H}+\vec{h}\right),$$

$$\nabla.\vec{q}=0$$

$$E\frac{\partial(T+\theta)}{\partial t}+(\vec{q}.\nabla)(T+\theta)=\kappa\nabla^2(T+\theta),$$

$$E'\frac{\partial(C+\gamma)}{\partial t}+(\vec{q}.\nabla)(C+\gamma)=\kappa'\nabla^2(C+\gamma),$$

$$\nabla.(\vec{H}+\vec{h})=0,$$

$$\in\frac{\partial(\vec{H}+\vec{h})}{\partial t}=\left[\left(\vec{H}+\vec{h}\right)\nabla\right]\vec{q}+\in\eta\nabla^2(\vec{H}+\vec{h})$$

Therefore, using initial conditions (8), retaining only linear terms, the linearized hydromagnetic perturbation equations are

$$\frac{1}{\in}\frac{\partial\vec{q}}{\partial t}=-\frac{1}{\rho_o}\nabla\delta p+\vec{g}\frac{\delta\rho}{\rho_o}-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\vec{q}+\frac{\mu_e}{4\pi\rho_o}\left(\nabla\times\vec{h}\right)\times\vec{H}, \tag{9}$$

$$\nabla.\vec{q}=0 \tag{10}$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta, \tag{11}$$

$$E'\frac{\partial\gamma}{\partial t}=\beta' w+\kappa'\nabla^2\gamma, \tag{12}$$

$$\nabla.\vec{h}=0, \tag{13}$$

$$\in\frac{\partial\vec{h}}{\partial t}=\left(\vec{H}.\nabla\right)\vec{q}+\in\eta\nabla^2\vec{h} \tag{14}$$

Equation of state, after perturbation, becomes

$$\rho+\delta\rho=\rho_o[1-\alpha(T+\theta-T_o)+\alpha'(C+\gamma-C_o)],$$

$$\rho+\delta\rho=\rho_o[1-\alpha(T-T_o)+\alpha'(C-C_o)]-\alpha\rho_o\theta+\alpha'\rho_o\gamma$$

$$\therefore\ \delta\rho=-\rho_o(\alpha\theta-\alpha'\gamma). \tag{15}$$

Writing the equations (9) – (14) in scalar form, we have

$$\frac{1}{\in}\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_o}\right)-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]u+\frac{\mu_e H}{4\pi\rho_o}\left[\frac{\partial h_x}{\partial z}-\frac{\partial h_z}{\partial x}\right], \tag{16}$$

$$\frac{1}{\in}\frac{\partial v}{\partial t}=-\frac{\partial}{\partial y}\left(\frac{\delta p}{\rho_o}\right)-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]v+\frac{\mu_e H}{4\pi\rho_o}\left[\frac{\partial h_y}{\partial z}-\frac{\partial h_z}{\partial y}\right], \tag{17}$$

$$\frac{1}{\in}\frac{\partial w}{\partial t}=-\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_o}\right)-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]w+g(\alpha\theta-\alpha'\gamma), \tag{18}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{19}$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta, \tag{20}$$

$$E'\frac{\partial\gamma}{\partial t}=\beta' w+\kappa'\nabla^2\gamma, \tag{21}$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0, \tag{22}$$

$$\in\frac{\partial h_x}{\partial t}=H\frac{\partial u}{\partial z}+\in\eta\nabla^2 h_x, \tag{23}$$

$$\in\frac{\partial h_y}{\partial t}=H\frac{\partial v}{\partial z}+\in\eta\nabla^2 h_y, \tag{24}$$

$$\in\frac{\partial h_z}{\partial t}=H\frac{\partial w}{\partial z}+\in\eta\nabla^2 h_z, \tag{25}$$

Multiply the equations (16) and (17) by $-\frac{\partial}{\partial x}$ and $-\frac{\partial}{\partial y}$ and adding , we get

$$\frac{1}{\in}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial z}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\delta p}{\rho_o}-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\frac{\partial w}{\partial z}+\frac{\mu_e H}{4\pi\rho_o}\nabla^2 h_z \tag{26}$$

eliminating $\frac{\delta p}{\rho_0}$ between (18) and (26), we get

$$\frac{1}{\in}\frac{\partial}{\partial t}\nabla^2 w=g(\alpha\theta-\alpha'\gamma)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\nabla^2 w+\frac{\mu_e H}{4\pi\rho_o}\frac{\partial}{\partial z}\nabla^2 h_z \tag{27}$$

Also, equations (20) , (21) and (25) give

$$E\frac{\partial\theta}{\partial t} - \kappa\nabla^2\theta = \beta w, \tag{28}$$

$$E'\frac{\partial\gamma}{\partial t} - \kappa'\nabla^2\gamma = \beta' w, \tag{29}$$

$$\in\frac{\partial h_z}{\partial t} - \in\eta\nabla^2 h_z = H\frac{\partial w}{\partial z}, \tag{30}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

DISPERSION RELATION

We now analyse the disturbances into normal modes and assume that the perturbation quantities are of the form

$$[\, w, \theta, \gamma, h_z \,] = [\, W(z), \Theta(z), \Gamma(z), K(z) \,] \exp(\, ik_x x + ik_y y + nt \,) \tag{31}$$

where k_x and k_y are the wave numbers in the x-and y-directions respectively, $k = \left(k_x^2 + k_y^2\right)^{1/2}$ is the resultant wave number, and n is the growth rate which is, in general, a complex constant.

For functions with the dependence (31)

$$\frac{\partial}{\partial t} = n, \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2 \text{ and } \nabla^2 = \frac{d^2}{dz^2} - k^2,$$

equations (27) – (30) become

$$\frac{n}{\in}\left[\frac{d^2}{dz^2} - k^2\right]W = -g(\alpha\Theta - \alpha'\Gamma)k^2 - \frac{1}{k_1}[\nu + \nu' n]\left[\frac{d^2}{dz^2} - k^2\right]W + \frac{\mu_e H}{4\pi\rho_o}\frac{d}{dz}\left[\frac{d^2}{dz^2} - k^2\right]K \tag{32}$$

$$En\Theta - \kappa\left[\frac{d^2}{dz^2} - k^2\right]\Theta = \beta W \tag{33}$$

$$E' n\Gamma - \kappa'\left[\frac{d^2}{dz^2} - k^2\right]\Gamma = \beta' W \tag{34}$$

$$\in n K - \in \eta \left[\frac{d^2}{dz^2} - k^2 \right] K = H \frac{dW}{dz} \tag{35}$$

Expressing the coordinates x, y, z in the new unit of length d i.e. $x = x'd$, $y = y'd$, $z = z'd$ and letting $a = kd$, $\sigma = nd^2/\nu$, $p_1 = \nu/\kappa$, $p_2 = \nu/\eta$, $q = \nu/\kappa'$, $F = \nu'/k_1$, $P_l = k_1/d^2$, equations (32) –(35) become

$$\frac{\sigma\nu}{\in d^2}\left[\frac{D^2}{d^2} - \frac{a^2}{d^2}\right]W = -g(\alpha\Theta - \alpha'\Gamma)\frac{a^2}{d^2} - \frac{1}{k_1}[\nu + \nu' n]\left[\frac{D^2}{d^2} - \frac{a^2}{d^2}\right]W + \frac{\mu_0 HD}{4\pi\rho_o d}\left[\frac{D^2}{d^2} - \frac{a^2}{d^2}\right]K$$

$\Big[\because \sigma = nd^2/\nu \Rightarrow n = \sigma\nu/d^2$ and $z = z'd, (dz)^2 = d^2(dz')^2$ and $D = d/dz'$

therefore $\dfrac{d^2}{dz^2} = \dfrac{d^2}{d^2(dz')^2} = \left(\dfrac{d}{dz'}\right)^2 \dfrac{1}{d^2} = \dfrac{D^2}{d^2}\Big]$

$$\frac{\sigma\nu}{\in d^4}[D^2 - a^2]W + \frac{1}{k_1 d^2}[\nu + \nu' n](D^2 - a^2)W = -\frac{ga^2}{d^2}(\alpha\Theta - \alpha'\Gamma) + \frac{\mu_0 HD}{4\pi\rho_o d^3}[D^2 - a^2]K$$

$$\frac{\sigma\nu}{\in d^4}[D^2 - a^2]W + \frac{\nu}{k_1 d^2}(D^2 - a^2)W + \frac{\nu' n}{k_1 d^2}(D^2 - a^2)W = -\frac{ga^2}{d^2}(\alpha\Theta - \alpha'\Gamma) + \frac{\mu_0 HD}{4\pi\rho_o d^3}[D^2 - a^2]K$$

Dividing each term by d^4/ν ,we get

$$\frac{\sigma}{\in}[D^2 - a^2]W + \frac{d^2}{k_1}(D^2 - a^2)W + \frac{\nu' d^2 n}{k_1 \nu}(D^2 - a^2)W = -\frac{ga^2 d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) + \frac{\mu_0 Hd}{4\pi\rho_o \nu}[D^2 - a^2]DK$$

$$\frac{\sigma}{\in}[D^2 - a^2]W + \frac{1}{P_l}(D^2 - a^2)W + \sigma F(D^2 - a^2)W = -\frac{ga^2 d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) + \frac{\mu_0 Hd}{4\pi\rho_o \nu}[D^2 - a^2]DK$$

$$\left[\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right](D^2 - a^2)W + \frac{ga^2 d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) - \frac{\mu_e Hd}{4\pi\rho_o \nu}[D^2 - a^2]DK = 0 \tag{36}$$

Now

$$E n\Theta - \kappa\left[\frac{d^2}{dz^2} - k^2\right]\Theta = \beta W \quad \text{becomes}$$

$$E\frac{\sigma\nu}{d^2}\Theta - \kappa\left[\frac{D^2}{d^2} - \frac{a^2}{d^2}\right]\Theta = \beta W$$

Or $E\frac{\sigma\nu}{d^2}\Theta - \frac{\kappa}{d^2}\left(D^2 - a^2\right)\Theta = \beta W$

Or $\frac{\left[E\sigma\nu - \kappa D^2 + \kappa a^2\right]\Theta}{d^2} = \beta W$

Or $\left[E\sigma p_1\kappa - \kappa D^2 + \kappa a^2\right]\Theta = \beta W d^2$

Or $-\kappa\left[D^2 - a^2 - E\sigma p_1\right]\Theta = \beta W d^2$

$$\left[D^2 - a^2 - E\sigma p_1\right]\Theta = -\left(\beta\frac{d^2}{\kappa}\right)W. \qquad (37)$$

and $E' n\,\Gamma - \kappa'\left[\frac{d^2}{dz^2} - k^2\right]\Gamma = \beta' w$ becomes

$$E'\frac{\sigma\nu}{d^2}\Gamma - \kappa'\left[\frac{D^2}{d^2} - \frac{a^2}{d^2}\right]\Gamma = \beta' W$$

Or $\left[E'\frac{\sigma\nu}{d^2} - \frac{\kappa'}{d^2}\left(D^2 - a^2\right)\right]\Gamma = \beta' W$

Or $\frac{\left(E'\sigma\nu - \kappa' D^2 + \kappa' a^2\right)\Gamma}{d^2} = \beta' W$

Or $\left(E'\sigma\nu - \kappa' D^2 + \kappa' a^2\right)\Gamma = \beta' W d^2$

$$-\kappa'\left(D^2 - a^2 - E' q\sigma\right)\Gamma = \beta' W d^2$$

$$\left[D^2 - a^2 - E' q\sigma\right]\Gamma = -\left(\beta'\frac{d^2}{\kappa'}\right)W \qquad (38)$$

and $\in n\,K - \in\eta\left[\frac{d^2}{dz^2} - k^2\right]K = H\frac{dW}{dz}$ becomes

$$\in\frac{\sigma\nu}{d^2}K - \in\eta\left[\frac{D^2}{d^2} - \frac{a^2}{d^2}\right]K = H\frac{DW}{d}$$

or $\in\frac{\sigma\nu}{d^2}K - \frac{\in\eta}{d^2}\left(D^2 - a^2\right)K = H\frac{DW}{d}$

or $$\frac{\left(\in \sigma\nu - \in \eta D^2 + \in \eta a^2\right)K}{d^2} = H\frac{DW}{d}$$

or $$\left[\in \sigma p_2 \eta - \in \eta D^2 + \in \eta a^2\right] K = HdDW$$

or $$-\in \eta\left[D^2 - a^2 - p_2\sigma\right] K = HdDW$$

or $$\left[D^2 - a^2 - p_2\sigma\right]K = -\left(\frac{Hd}{\in \eta}\right)DW. \tag{39}$$

Applying the operator $\left[D^2 - a^2 - p_2\sigma\right]\left(D^2 - a^2 - Ep_1\sigma\right)\left(D^2 - a^2 - E'q\sigma\right)$ to the equation (36) and using equations (37) – (39); thus eliminating Θ, K, Γ, we obtain

$$\begin{aligned}
&\left(D^2 - a^2\right)\left(D^2 - a^2 - p_2\sigma\right)\left(D^2 - a^2 - Ep_1\sigma\right)\left(D^2 - a^2 - E'q\sigma\right)\left[\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right]W \\
&- Ra^2W\left(D^2 - a^2 - p_2\sigma\right)\left(D^2 - a^2 - E'q\sigma\right) + Sa^2\left(D^2 - a^2 - p_2\sigma\right)\left(D^2 - a^2 - Ep_1\sigma\right)W \\
&+ Q\left(D^2 - a^2\right)\left(D^2 - a^2 - Ep_1\sigma\right)\left(D^2 - a^2 - E'q\sigma\right)D^2W = 0
\end{aligned} \tag{40}$$

where $R = \frac{g\alpha\beta d^4}{\nu\kappa}$ is the Rayleigh number

$S = \frac{g\alpha'\beta' d^4}{\nu\kappa'}$ is the analogous solute Rayleigh number and

$Q = \frac{\mu_e H^2 d^2}{4\pi\rho_o \nu\eta \in}$ is the Chandrasekhar number

Now consider the case when both boundaries are free as well as maintained at constant temperature and solute concentrations, while the adjoining medium is perfectly conducting. The boundary conditions appropriate to the problem are

$$\left.\begin{array}{l}
W = D^2W = 0,\ \Theta = \Gamma = 0 \text{ at } z = 0 \text{ and } 1 \\
K = 0 \text{ on a perfectly conducting boundary and} \\
h_x,\ h_y,\ h_z \text{ are continuous with an external vacuum field on a non-conducting boundary.}
\end{array}\right\} \tag{41}$$

The case of two free boundaries is a little artificial but it enables us to find analytical solutions and to make some qualitative conclusions.

Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for z = 0 and 1 and hence the proper solution of W characterizing the lowest mode is :

$W = W_o \sin \pi z,$ Where W_o is a constant

So, $DW = \pi W_o \cos \pi z$

and $D^2W = -\pi^2 W_o \sin \pi z$

or $D^2W = -\pi^2 W$

$$\therefore \quad D^2 = -\pi^2 \tag{42}$$

substituting this value of D^2 in equation (40), we get

$$\begin{aligned}&\left(\pi^2 + a^2\right)\left(\pi^2 + a^2 + p_2\sigma\right)\left(\pi^2 + a^2 + Ep_1\sigma\right)\left(\pi^2 + a^2 + E'q\sigma\right)\left[\frac{\sigma}{\epsilon} + \frac{1}{P_l} + \sigma F\right] \\ &- Ra^2\left(\pi^2 + a^2 + p_2\sigma\right)\left(\pi^2 + a^2 + E'q\sigma\right) + Sa^2\left(\pi^2 + a^2 + p_2\sigma\right)\left(\pi^2 + a^2 + Ep_1\sigma\right) \\ &+ Q\left(\pi^2 + a^2\right)\left(\pi^2 + a^2 + Ep_1\sigma\right)\left(\pi^2 + a^2 + E'q\sigma\right)\pi^2 = 0\end{aligned} \tag{43}$$

putting $x = a^2/\pi^2$, $R_1 = R/\pi^4$, $Q_1 = Q/\pi^2$, $P_l = P/\pi^2$, $S_1 = S/\pi^4$ and $i\sigma_1 = \sigma/\pi^2$,

it being remembered that σ can be complex, equation (43) becomes

$$\begin{aligned}&(1 + x)(1 + x + ip_2\sigma_1)(1 + x + iEp_1\sigma_1)\left(1 + x + iE'q\sigma_1\right)\left[\frac{i\sigma_1}{\epsilon} + \frac{1}{P} + i\sigma_1 F\right] \\ &- R_1 x(1 + x + ip_2\sigma_1)\left(1 + x + iE'q\sigma_1\right) + S_1 x(1 + x + ip_2\sigma_1)(1 + x + iEp_1\sigma_1) \\ &+ Q(1 + x)(1 + x + iEp_1\sigma_1)\left(1 + x + iE'q\sigma_1\right) = 0\end{aligned}$$

or

$$R_1 = \left(\frac{1 + x}{x}\right)\left[\frac{i\sigma_1}{\epsilon} + \frac{1}{P} + i\sigma_1 F\right][1 + x + iEp_1\sigma_1] + Q_1\left(\frac{1 + x}{x}\right)\frac{(1 + x + iEp_1\sigma_1)}{(1 + x + ip_2\sigma_1)} + S_1\frac{(1 + x + iEp_1\sigma_1)}{\left(1 + x + iE'q\sigma_1\right)} \tag{44}$$

Equation (44) is the required dispersion relation including the effects of magnetic field, medium permeability, kinematic viscoelasticity and stable solute gradient on thermosolutal instability of Rivlin-Ericksen fluid in porous medium in hydromagnetics.

THE STATIONARY CONVECTION

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (44) reduces to

$$R_1 = \frac{(1+x)^2}{xP} + Q_1 \frac{1+x}{x} + S_1, \tag{45}$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters S_1, Q_1 and P. The parameter F accounting for the kinematic viscoelasticity effect vanishes for the stationary convection.

To investigate the effects of stable solute gradient, magnetic field and medium permeability, we examine the behavior of dR_1/dS_1, dR_1/ dQ_1 and dR_1/dP analytically. Equation (45) gives.

$$\frac{dR_1}{dS_1} = 1, \tag{46}$$

$$\frac{dR_1}{dQ_1} = \frac{1+x}{x}, \tag{47}$$

$$\frac{dR_1}{dP} = -\frac{(1+x)^2}{xP^2}. \tag{48}$$

Thus for stationary convection, the stable solute gradient and magnetic field are found to have stabilizing effects, whereas, the medium permeability has a destabilizing effect on thermosolutal instability of Rivlin-Ericksen fluid in porous medium in hydromagnetics.

STABILITY OF THE SYSTEM AND OSCILLATORY MODES

Multiplying equation (36) by W^* (complex conjugate of W) and integrating over the range of z and also using (37) – (39) together with the boundary conditions (41) we obtain

$$\left(\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right)\int_0^1 W^*(D^2 - a^2)W dz + \frac{g\alpha a^2 d^2}{\nu}\int_0^1 \Theta W^* dz - \frac{g\alpha' a^2 d^2}{\nu}\int_0^1 \Gamma W^* dz$$
$$- \frac{\mu_e H d}{4\pi\rho_o \nu}\int_0^1 W^*(D^2 - a^2)DK = 0,$$

$$\left(\frac{\sigma}{\in}+\frac{1}{P_l}+\sigma F\right)\left\{\left[(W^{*}DW)\right]_0^1-\int_0^1 DW^{*}DWdz-a^2\int_0^1 WW^{*}dz\right\}+\frac{g\alpha a^2 d^2}{\nu}$$

$$\int_0^1\Theta\left(\frac{-\kappa}{\beta d^2}\right)\left(D^2-a^2-Ep_1\sigma^{*}\right)\Theta^{*}dz-\frac{g\alpha' a^2 d^2}{\nu}\int_0^1\Gamma\left(\frac{-\kappa'}{\beta' d^2}\right)\left(D^2-a^2-Ep_1\sigma^{*}\right)\Gamma^{*}dz$$

$$-\frac{\mu_e Hd}{4\pi\rho_o\nu}\left\{\left[W^{*}\left(D^2-a^2\right)K\right]_0^1-\int_0^1 DW^{*}\left(D^2-a^2\right)Kdz\right\}=0,$$

or

$$-\left(\frac{\sigma}{\in}+\frac{1}{P_l}+\sigma F\right)\int_0^1\left\{|DW|^2+a^2|W|^2\right\}dz-\frac{g\alpha a^2\kappa}{\nu\beta}\int_0^1\Theta\left(D^2-a^2-Ep_1\sigma^{*}\right)\Theta^{*}dz$$

$$+\frac{g\alpha'\kappa' a^2}{\nu\beta'}\int_0^1\Gamma\left(D^2-a^2-E'p_1\sigma^{*}\right)\Gamma^{*}dz+\frac{\mu_e Hd}{4\pi\rho_o\nu}\int_0^1\left(-\frac{\in\eta}{Hd}\right)\left(D^2-a^2-p_2\sigma^{*}\right)K^{*}\left(D^2-a^2\right)Kdz=0$$

or

$$-\left(\frac{\sigma}{\in}+\frac{1}{P_l}+\sigma F\right)\int_0^1\left\{|DW|^2+a^2|W|^2\right\}dz-\frac{g\alpha a^2\kappa}{\nu\beta}\left\{\left[\Theta D\Theta^{*}\right]_0^1-\int_0^1 D\Theta D\Theta^{*}dz-a^2\int_0^1\Theta\Theta^{*}dz\right.$$

$$\left.-Ep_1\sigma^{*}\int_0^1\Theta\Theta^{*}dz\right\}+\frac{g\alpha'\kappa' a^2}{\nu\beta'}\left\{\left[\Gamma D\Gamma^{*}\right]_0^1-\int_0^1 D\Gamma D\Gamma^{*}dz-a^2\int_0^1\Gamma\Gamma^{*}dz-E'q\sigma^{*}\int_0^1\Gamma\Gamma^{*}dz\right\}$$

$$-\frac{\mu_e\in\eta}{4\pi\rho_o\nu}\int_0^1\left(D^2K^{*}-a^2K^{*}\right)\left(D^2K-a^2K\right)dz-p_2\sigma^{*}\left[K^{*}DK\right]_0^1-\int_0^1 DK^{*}DKdz-a^2\int_0^1 K^{*}Kdz=0$$

where the integrated part vanishes each time on account of the boundary conditions (41)

$$\left(\frac{\sigma}{\in}+\frac{1}{P_l}+\sigma F\right)I_1-\frac{g\alpha a^2\kappa}{\nu\beta}\left[I_2+Ep_1\sigma^{*}I_3\right]+\frac{g\alpha'\kappa' a^2}{\nu\beta'}\left[I_4+E'q\sigma^{*}I_5\right]+\frac{\mu_e\in\eta}{4\pi\rho_o\nu}\left[I_6+p_2\sigma^{*}I_7\right]=0 \tag{49}$$

where

$$I_1=\int_0^1\left\{|DW|^2+a^2|W|^2\right\}dz$$

$$I_2=\int_0^1\left\{|D\Theta|^2+a^2|\Theta|^2\right\}dz,\; I_3=\int_0^1|\Theta|^2dz,$$

$$I_4=\int_0^1\left\{|D\Gamma|^2+a^2|\Gamma|^2\right\}dz,\; I_5=\int_0^1|\Gamma|^2dz,$$

$$I_6 = \int_0^1 \left\{ |D^2K|^2 + 2a^2 |DK|^2 + a^4 |K|^2 \right\} dz$$

And

$$I_7 = \int_0^1 \left\{ |DK|^2 + a^2 |K|^2 \right\} dz$$

The integral $I_1 \ldots I_7$ are all positive definite. Putting $\sigma = \sigma_r + i\sigma_i$ in equation (49) and equating real and imaginary parts of equation (49), we obtain

$$\left[\left(\frac{1}{\in} + F\right) I_1 - \frac{g\alpha a^2 \kappa}{\nu\beta} E p_1 I_3 + \frac{g\alpha' \kappa' a^2}{\nu\beta'} E' q I_5 + \frac{\mu_e \in \eta}{4\pi\rho_o \nu} p_2 I_7\right]\sigma_r = -\left[\frac{I_1}{P_l} - \frac{g\alpha a^2 \kappa}{\nu\beta} I_2 + \frac{g\alpha' \kappa' a^2}{\nu\beta'} I_4 + \frac{\mu_e \in \eta}{4\pi\rho_o \nu} I_6\right] \tag{50}$$

and

$$\left[\left(\frac{1}{\in} + F\right) I_1 + \frac{g\alpha a^2 \kappa}{\nu\beta} E p_1 I_3 - \frac{g\alpha' \kappa' a^2}{\nu\beta'} E' q I_5 - \frac{\mu_e \in \eta}{4\pi\rho_o \nu} p_2 I_7\right]\sigma_i = 0 \tag{51}$$

It is evident from (50) that σ_r is positive or negative. The system is, therefore, stable or unstable. It is clear from (51) that σ_i may be zero or non-zero, meaning that the modes may be oscillatory or non-oscillatory. The oscillatory modes are introduces due to the presence of stable solute gradient and magnetic field, which were non-existent in their absence.

THE CASE OF OVERSTABILITY

We discuss the possibility of whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (44) will admit solutions with real σ_1. Equation (44) can be written as

$$R_1 = \frac{(1+x)\left[\frac{i\sigma_1}{\in}+\frac{1}{P}+i\sigma_1 F\right][1+x+iEp_1\sigma_1][1+x+ip_2\sigma_1][1+x+iE'q\sigma_1] + Q_1(1+x)[1+x+iEp_1\sigma_1][1+x+iE'q\sigma_1]+S_1x(1+x+iEp_1\sigma_1)(1+x+ip_2\sigma_1)}{x(1+x+ip_2\sigma_1)(1+x+iE'q\sigma_1)}$$

(52)

Putting $C_1 = \sigma_1^2$ and $b = (1+x)$ and eliminating R_1 between the real and imaginary parts of equation (52), we obtain

$$E'p_2^2q^2\left[b\left(\frac{1}{\in}+F\right)+E\frac{p_1}{P}\right]C_1^2+\left[b\left(\frac{1}{\in}+F\right)+E\frac{p_1}{P}\right]\left[E'^2q^2+p_2^2\right]b^2+\left[Q_1E'^2q^2(Ep_1-p_2)\right]b$$
$$+(b_1-1)S_1p_2^2(Ep_1-E'q)C_1+\left[b\left(\frac{1}{\in}+F\right)+E\frac{p_1}{P}\right]b^4+bQ_1(Ep_1-p_2)b^2$$
$$+(b-1)S_1(Ep_1-E'q)b^2=0,$$

or

$$A_2 C_1^2 + A_1 C_1 + A_o = 0\,, \qquad (53)$$

where $A_2 = E'^2q^2\, p_2^2\left[b\left(\frac{1}{\in}+F\right)+E\frac{p_1}{P}\right]$

$$A_1 = \left[b\left(\frac{1}{\in}+F\right)+E\frac{p_1}{P}\right][E'^2q^2+p_2^2]\,b^2+[Q_1\,E'^2q^2\,(Ep_1-p_2)b+(b-1)S_1\,p_2^2\,(Ep_1-E'q).$$

$$A_o = \left[b\left(\frac{1}{\in}+F\right)+E\frac{p_1}{P}\right]b^4+bQ_1(Ep_1-p_2)\,b^2+(b-1)S_1\,(Ep_1-E'q)\,b^2$$

Since σ_1 is real for overstability, both the values of $C_1(=\sigma_1^2)$ are positive, equation (53) is quadratic in C_1 and does not involve any of its roots to be positive if :

$$Ep_1 > p_2 \text{ and } Ep_1 > E'q, \qquad (54)$$

which imply that

$$\kappa < E\eta \text{ and } E'\kappa < E\kappa' \qquad (55)$$

Thus $\kappa < E\eta$ and $E'\kappa < E\kappa'$ are the sufficient conditions for the non-existence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

CHAPTER -III

Thermosolutal Instability of Rivlin-Ericksen rotating fluid in Porous medium

INTRODUCTION

The Theoretical and experimental results of the onset of thermal instability in a fluid layer under varying assumptions of hydrodynamics has been treated in detail by Chandrasekhar (1961) in his celebrated monograph. The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been considered by Veronis (1965). The physics is quite similar in the stellar case in that helium acts like salt in raising the density and in diffusing more slowly than heat. The conditions under which convective motions are important in stellar atmospheres are usually far removed from consideration of single component fluid and rigid boundaries and therefore, it is desirable to consider a fluid acted on by a solute gradient and free boundaries. The thermosolutal convection problems arise in oceanography, limnology and engineering.

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. The Rivlin-Ericksen (1955) fluid is one such fluid. Many research workers have paid their attention towards the study of Rivlin-Ericksen fluid. Joshi (1976) has discussed the viscoelastic Rivlin-Ericksen incompressible fluid under time-dependent pressure gradient. Sharma and Kumar (1996) have studied the thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid acted on by a uniform rotation and found that rotation has a stabilizing effect and introduces oscillatory modes in the system.

In all the above studies, the medium has been considered to be non-porous. When the fluid permeates a porous material, the gross effect is represented by the Darcy's Law. As a result of this macroscopic law, the usual viscous term in the equation of Rivlin-Ericksen fluid motion is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)q\right]$, where μ and μ' are the viscosity and viscoelasticity of the Rivlin-Ericksen fluid, k_1 is the medium permeability and q is the Darcian (filter) velocity of the fluid. The problem of thermosolutal convection in fluids in a porous medium is of importance in geophysics, soil sciences, ground water hydrology and astrophysics. In many astrophysical situations, the effect of rotation on thermosolutal convection in porous medium is also important.

Keeping in mind the importance in geophysics, soil physics, astrophysics, ground water hydrology and various applications mentioned above, the thermosolutal instability of a Rivlin-Ericksen fluid in porous medium in the presence of uniform vertical rotation has been considered in the present chapter.

PERTURBATION EQUATIONS

Here we consider an infinite, horizontal, incompressible Rivlin-Ericksen fluid layer of thickness d, heated and soluted from below so that, the temperatures, densities and solute concentrations at the bottom surface z = 0 are T_o, ρ_o , C_o and at the upper surface z = d are T_d, ρ_d and C_d respectively, and that a uniform temperature gradient β $\left(=|dT/dz|\right)$ and a uniform solute gradient β' $\left(=|dC/dz|\right)$ are maintained. The gravity field $\vec{g}$ (0 ,0, −g) and a uniform vertical rotation $\vec{\Omega}(0,0,\Omega)$ pervade the system. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity $\in$ and medium permeability k_1. The relevant hydrodynamic equations, under Boussinesq approximation, are

$$\frac{1}{\in}\left(\frac{\partial\vec{q}}{\partial t}+\frac{1}{\in}(\vec{q}\cdot\nabla)\vec{q}\right)=-\frac{\nabla p}{\rho_o}+\left(1+\frac{\delta\rho}{\rho_o}\right)\vec{g}-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\vec{q}+\frac{2}{\in}\left(\vec{q}\times\vec{\Omega}\right), \tag{1}$$

$$\nabla.\vec{q}=0, \tag{2}$$

$$E\frac{\partial T}{\partial t} + (\vec{q}.\nabla)T = \kappa\nabla^2 T, \quad (3)$$

$$E'\frac{\partial C}{\partial t} + (\vec{q}.\nabla)C = \kappa'\nabla^2 C, \quad (4)$$

and

$$\rho = \rho_o [1 - \alpha (T - T_o) + \alpha' (C - C_o)]. \quad (5)$$

here p, ρ, T,C, α, α', g and $\vec{q}$(u ,v, w) denote, respectively, the fluid pressure, density, temperature, solute concentration, thermal coefficient of expansion, an analogous solvent coefficient of expansion, gravitational acceleration and fluid velocity.

here the suffix zero refers to the values at the reference level z = 0. The kinematic viscosity ν, kinematic viscoelasticity ν', the thermal diffusivity κ and the solute diffusivity κ' are all assumed to be constant. $E = \epsilon + (1 - \epsilon)(\rho_s c_s / \rho_o c_i)$ is a constant and E' is a constant analogous to E but corresponding to solute rather than heat. ρ_s, c_s, ρ_o and c_i denote the density and heat capacity of solid material and fluid, respectively.

Initially,

$\vec{q} = (0, 0, 0), \rho = \rho(z), p = p(z)$,

$T = -\beta z + T_o$,

$C = -\beta' z + C_o$,

and

$$\rho = \rho_o [1 + \alpha \beta z - \alpha'\beta' z].$$

Let δp, $\delta\rho$, θ, γ and $\vec{q}$(u ,v, w) denote, respectively, the perturbation in pressure p, density ρ, temperature T, solute concentration C and velocity $\vec{q}$(0 ,0, 0) .

So the change in density $\delta\rho$, caused by the perturbations θ and γ in temperature and solute concentration, is given by

$$\rho + \delta\rho = \rho_o [1 - \alpha (T + \theta - T_o) + \alpha' (C + \gamma - C_o)],$$

or $\rho + \delta\rho = \rho_o [1 - \alpha (T - T_o) + \alpha' (C - C_o)] - \alpha\rho_o\theta + \alpha' \rho_o\gamma$,

$$\therefore \delta\rho = -\rho_o (\alpha\theta - \alpha'\gamma). \tag{6}$$

Substituting the perturbed quantities in equations (1)-(4), we have

$$\frac{1}{\in}\left(\frac{\partial\vec{q}}{\partial t} + \frac{1}{\in}(\vec{q}\cdot\nabla)\vec{q}\right) = -\frac{\nabla(p+\delta p)}{\rho_o} + \left(1 + \frac{\delta\rho}{\rho_o}\right)\vec{g} - \frac{1}{k_1}\left[\nu + \nu'\frac{\partial}{\partial t}\right]\vec{q} + \frac{2}{\in}\left(\vec{q}\times\vec{\Omega}\right),$$

$$\nabla.\vec{q} = 0$$

$$E\frac{\partial(T+\theta)}{\partial t} + (\vec{q}.\nabla)(T+\theta) = \kappa\nabla^2(T+\theta),$$

$$E'\frac{\partial(C+\gamma)}{\partial t} + (\vec{q}.\nabla)(C+\gamma) = \kappa'\nabla^2(C+\gamma).$$

Using initial conditions and retaining only linear terms, the linearized hydrodynamic perturbation equations are

$$\frac{1}{\in}\frac{\partial\vec{q}}{\partial t} = -\frac{1}{\rho_o}(\nabla\delta p) - \vec{g}(\alpha\theta - \alpha'\gamma) - \frac{1}{k_1}\left[\nu + \nu'\frac{\partial}{\partial t}\right]\vec{q} + \frac{2}{\in}\left(\vec{q}\times\vec{\Omega}\right), \tag{7}$$

$$\nabla.\vec{q} = 0, \tag{8}$$

$$E\frac{\partial\theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \tag{9}$$

$$E'\frac{\partial\gamma}{\partial t} = \beta' w + \kappa'\nabla^2\gamma, \tag{10}$$

Writing equations (7) – (10) in scalar form, we have

$$\frac{1}{\in}\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_o}\right) - \frac{1}{k_1}\left[\nu + \nu'\frac{\partial}{\partial t}\right]u + \frac{2\Omega}{\in}v, \tag{11}$$

$$\frac{1}{\in}\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}\left(\frac{\delta p}{\rho_o}\right) - \frac{1}{k_1}\left[\nu + \nu'\frac{\partial}{\partial t}\right]v - \frac{2\Omega}{\in}u, \tag{12}$$

$$\frac{1}{\in}\frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_o}\right) - \frac{1}{k_1}\left[\nu + \nu'\frac{\partial}{\partial t}\right]w + g(\alpha\theta - \alpha'\gamma), \tag{13}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{14}$$

$$E\frac{\partial \theta}{\partial t}=\beta w+\kappa\nabla^2\theta, \tag{15}$$

$$E'\frac{\partial \gamma}{\partial t}=\beta' w+\kappa'\nabla^2\gamma \tag{16}$$

Eliminating $\frac{\delta p}{\rho_0}$ between (11)-(13) and using (14), we obtain

$$\frac{1}{\in}\frac{\partial}{\partial t}\nabla^2 w=g(\alpha\theta-\alpha'\gamma)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\frac{2\Omega}{\in}\frac{\partial\zeta}{\partial z}-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\nabla^2 w \tag{17}$$

Operating equations(12) by $\frac{\partial}{\partial x}$ and (11) by $-\frac{\partial}{\partial y}$ and adding , we obtain

$$\frac{1}{\in}\frac{\partial\zeta}{\partial t}=-\frac{1}{k_1}\left[\nu+\nu'\frac{\partial}{\partial t}\right]\zeta+\frac{2\Omega}{\in}\frac{\partial w}{\partial z} \tag{18}$$

where $\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

denotes the z-component of vorticity.

DISPERSION RELATION

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[\, w, \theta, \gamma, \zeta\,] = [\, W(z), \Theta(z), \Gamma(z), Z(z)\,] \exp(ik_x x + ik_y y + nt), \tag{19}$$

where k_x and k_y are the wave numbers in the x-and y-directions respectively, $k=\left(k_x^2+k_y^2\right)^{1/2}$ is the resultant wave number, and n is the growth rate which is, in general, a complex constant.

For functions with the dependence (19)

$$\frac{\partial}{\partial t}=n, \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=-k^2 \quad \text{and} \quad \nabla^2=\frac{d^2}{dz^2}-k^2$$

equations (15) – (18) become

$$En\Theta = \beta\, W + \kappa\left[\frac{d^2}{dz^2} - k^2\right]\Theta, \tag{20}$$

$$E'\, n\, \Gamma = \beta' W + \kappa'\left[\frac{d^2}{dz^2} - k^2\right]\Gamma, \tag{21}$$

$$\frac{1}{\in} n\left[\frac{d^2}{dz^2} - k^2\right]W = -g(\alpha\Theta - \alpha'\Gamma)k^2 - \frac{2\Omega}{\in}\frac{dZ}{dz} - \frac{1}{k_1}[\nu + \nu' n]\left[\frac{d^2}{dz^2} - k^2\right]W, \tag{22}$$

$$\frac{1}{\in} nZ = -\frac{1}{k_1}[\nu + \nu' n] + \frac{2\Omega}{\in}\frac{dW}{dz}. \tag{23}$$

Expressing the coordinates x, y, z in the new unit of length d i.e. $x = x'd$, $y = y'd$ and $z = z'd$

and letting

$a = kd$, $\sigma = nd^2/\nu$, $p_1 = \nu/k$, $q = \nu/\kappa'$, $F = \nu'/k_1$, $P_l = k_1/d^2$, equations (20)-(23) become

$$[D^2 - a^2 - E\sigma p_1]\Theta = -\left(\beta\frac{d^2}{\kappa}\right)W, \tag{24}$$

$$[D^2 - a^2 - E' q\sigma]\Gamma = -\left(\beta'\frac{d^2}{\kappa'}\right)W, \tag{25}$$

$$\left[\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right](D^2 - a^2)W + \frac{ga^2d^2}{\nu}(\alpha\Theta - \alpha'\Gamma) + \frac{2\Omega d^3}{\in\nu}DZ = 0, \tag{26}$$

$$\left[\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right]Z = \left(\frac{2\Omega d}{\in\nu}\right)DW \tag{27}$$

Eliminating Θ, Γ and Z between equations (24)-(27), we get

$$\left[\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right](D^2 - a^2 - Ep_1\sigma)(D^2 - a^2)W + Sa^2\frac{(D^2 - a^2 - Ep_1\sigma)}{(D^2 - a^2 - E'q\sigma)}W + \frac{T_A(D^2 - a^2 - Ep_1\sigma)}{\left(\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right)}D^2W = Ra^2W \tag{28}$$

where $R = \frac{g\alpha\beta d^4}{\nu\kappa}$, $S = \frac{g\alpha'\beta' d^4}{\nu\kappa'}$, $T_A = \frac{4\Omega^2 d^4}{\in^2 \nu^2}$.

Now consider the case where both boundaries are free as well as maintained at constant temperatures and solute concentrations.

The boundary conditions appropriate to the problem are

$$W = D^2W = 0,\ \Theta = 0,\ \Gamma = 0,\ DZ = 0 \text{ at } z = 0 \text{ and } 1. \tag{29}$$

The case of two free boundaries is a little artificial but it enables us to find analytical solutions and to make some qualitative conclusions. This is the most appropriate for stellar atmospheres. Using the above boundary conditions, it can be shown that all the even order derivatives of W must vanish for z = 0 and 1 and hence the proper solution of W characterizing the lowest mode is

$$W = W_o \sin \pi z, \tag{30}$$

where W_o is a constant.

$$\therefore \quad DW = \pi W_o \cos \pi z$$

$$\text{and} \quad D^2W = -\pi^2 W_o \sin \pi z$$

$$D^2W = -\pi^2 W$$

$$\text{or} \quad D^2 = -\pi^2 \tag{31}$$

substituting this value of D^2 in equation (28), we get

$$\left[\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right](\pi^2 + a^2 + Ep_1\sigma)(\pi^2 + a^2) + Sa^2\frac{(\pi^2 + a^2 + Ep_1\sigma)}{(\pi^2 + a^2 + E'q\sigma)} + \frac{T_A(\pi^2 + a^2 + Ep_1\sigma)\pi^2}{\left(\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right)} = Ra^2$$

letting $x = \frac{a^2}{\pi^2}$, $i\sigma_1 = \frac{\sigma}{\pi^2}$, $P = \pi^2 P_l$, $R_1 = \frac{R}{\pi^4}$, $S_1 = \frac{S}{\pi^4}$, $T_{A_1} = \frac{T}{\pi^4}$ we get

$$R_1 = \frac{1+x}{x}\left[\frac{i\sigma_1}{\in} + \frac{1}{P} + i\sigma_1 F\right]\left[1 + x + iEp_1\sigma_1\right] + S_1 \frac{(1 + x + iEp_1\sigma_1)}{(1 + x + iE'q\sigma_1)} + T_{A_1} \frac{(1 + x + iEp_1\sigma_1)}{x\left[\frac{i\sigma_1}{\in} + \frac{1}{P} + i\sigma_1 F\right]}. \quad (32)$$

Equation (32) is the required dispersion relation studying the effects of rotation, medium permeability, kinematic viscoelasticity and stable solute gradient on thermosolutal instability of Rivlin-Ericksen rotating fluid in porous medium.

THE STATIONARY CONVECTION

When the instability sets in as stationary convection, the marginal state will be characterized by $\sigma = 0$. Putting $\sigma = 0$, the dispersion relation (32) reduces to

$$R_1 = \frac{(1+x)^2}{xP} + T_{A_1} P \frac{(1+x)}{x} + S_1, \quad (33)$$

which expresses the modified Rayleigh number R_1 as a function of the dimensionless wave number x and the parameters S_1, T_{A_1} and P. The parameter F accounting for the viscoelasticity effect disappears for the stationary convection.

To investigate the effects of stable solute gradient, rotation and medium permeability, we examine the behaviour of dR_1/dS_1 , dR_1/dT_{A_1} and dR_1/dP analytically. Equation (33) yields

$$\frac{dR_1}{dS_1} = 1, \quad (34)$$

which implies that the stable solute gradient has a stabilizing effect on the thermosolutal convection. The adverse solute gradient has destabilizing effect on the system since dR_1/dS_1 then becomes negative. Equation (33) also yields

$$\frac{dR_1}{dT_{A_1}} = \left(\frac{1+x}{x}\right)P, \quad (35)$$

The rotation, therefore, has always a stabilizing effect on the thermosolutal instability of Rivlin-Ericksen rotating field in porous medium.

It is evident from (33) that

$$\frac{dR_1}{dP} = -\left(\frac{1+x}{x}\right)\left[\frac{(1+x)}{P^2} - T_{A_1}\right]. \tag{36}$$

In the absence of rotation ($T_{A_1} \to 0$), $\frac{dR_1}{dP}$ is given by

$$\frac{dR_1}{dP} = -\left[\frac{(1+x)^2}{xP^2}\right] \tag{37}$$

which is always negative. The medium permeability, therefore, has a destabilizing effect on the thermosolutal instability of a fluid in the absence of rotation. In the presence of rotation, the system is unstable or stable if

$$T_{A_1} < (\text{or} >) \frac{1+x}{P^2}. \tag{38}$$

STABILITY OF THE SYSTEM AND OSCILLATORY MODES

Multiplying equation (26) by W^*, the complex conjugate of W, and using (24),(25),(27) together with the boundary conditions (29) and also integrating over the range of z, we obtain

$$\left(\frac{\sigma}{\in} + \frac{1}{P_l} + \sigma F\right)I_1 - \frac{g\alpha a^2\kappa}{\nu\beta}\left[I_2 + Ep_1\sigma^* I_3\right] + \frac{g\alpha'\kappa' a^2}{\nu\beta'}\left[I_4 + E'q\sigma^* I_5\right] + d^2\left(\frac{\sigma^*}{\in} + \frac{1}{P_l} + \sigma^* F\right)I_6 = 0, \tag{39}$$

where

$$I_1 = \int_0^1 \left\{ |DW|^2 + a^2 |W|^2 \right\} dz$$

$$I_2 = \int_0^1 \left\{ |D\Theta|^2 + a^2 |\Theta|^2 \right\} dz,$$

$$I_3 = \int_0^1 |\Theta|^2 \, dz,$$

$$I_4 = \int_0^1 \left\{ |D\Gamma|^2 + a^2 |\Gamma|^2 \right\} dz,$$

$$I_5 = \int_0^1 |\Gamma|^2 \, dz,$$

$$I_6 = \int_0^1 |Z|^2 \, dz,$$

and integrated part vanishes each time on account of boundary conditions (29).

The integrals $I_1 \ldots I_6$ are all positive definite. Putting $\sigma = \sigma_r + i\sigma_i$ and equating the real and imaginary parts of (39), we obtain

$$\left[\left(\frac{1}{\epsilon} + F\right) I_1 + \frac{g\alpha'\kappa' a^2}{\nu\beta'} E' q I_5 + d^2 \left(\frac{1}{\epsilon} + F\right) I_6 - \frac{g\alpha a^2 \kappa}{\nu\beta} [E p_1 I_3]\right] \sigma_r = -\left[\frac{I_1}{P_l} - \frac{g\alpha a^2 \kappa}{\nu\beta} I_2 + \frac{g\alpha'\kappa' a^2}{\nu\beta'} I_4 + \frac{d^2}{P_l} I_6\right] \tag{40}$$

and

$$\left[\left(\frac{1}{\epsilon} + F\right) I_1 - \frac{g\alpha'\kappa' a^2}{\nu\beta'} E' q I_5 - d^2 \left(\frac{1}{\epsilon} + F\right) I_6 + \frac{g\alpha\kappa a^2}{\nu\beta} E p_1 I_3\right] \sigma_i = 0 \tag{41}$$

It is evident from (40) that σ_r is positive or negative. The system is, therefore, stable or unstable. It is clear from (41) that σ_i may be zero or non-zero, meaning that the modes may be oscillatory or non-oscillatory. The oscillatory modes are introduced due to the presence of rotation, stable gradient and viscoelasticity, which were non-existent in their absence.

THE CASE OF OVERSTABILITY

We discuss the possibility of whether instability may occur as overstability. Since we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it suffices to find conditions for which (32) will admit solutions with σ_1 real.

Equating real and imaginary parts of equation (32), we have

$$R_1x\left[\frac{1+x}{P}-\sigma_1^2E'q\left(\frac{1}{\in}+F\right)\right]=$$

$$\left[-\sigma_1^2\left(\frac{1}{\in}+F\right)^2(1+x)^3-2\sigma_1^2E'q\left(\frac{1}{\in}+F\right)\frac{(1+x)^2}{P}+\sigma_1^4EE'p_1q\left(\frac{1}{\in}+F\right)^2(1+x)\right.$$

$$-2\sigma_1^2Ep_1\left(\frac{1}{\in}+F\right)\frac{(1+x)^2}{P}+\frac{(1+x)^3}{P^2}-\sigma_1^2EE'p_1q\frac{(1+x)}{P^2}+T_{A_1}(1+x)^2-T_{A_1}EE'p_1q\sigma_1^2$$

$$\left.+S_1x\frac{(1+x)}{P}-S_1x\sigma_1^2Ep_1\left(\frac{1}{\in}+F\right)\right]$$

and

$$iR_1x\left[\sigma_1\left(\frac{1}{\in}+F\right)(1+x)+E'\frac{q\sigma_1}{P}\right]=$$

$$i\left[-\sigma_1^3E'q\left(\frac{1}{\in}+F\right)^2(1+x)^2+2\sigma_1\left(\frac{1}{\in}+F\right)\frac{(1+x)^3}{P}-\sigma_1^3Ep_1\left(\frac{1}{\in}+F\right)^2(1+x)^2\right.$$

$$-2\sigma_1^3EE'p_1q\left(\frac{1}{\in}+F\right)\frac{(1+x)}{P}+\sigma_1E'q\frac{(1+x)^2}{P^2}+\sigma_1Ep_1\frac{(1+x)^2}{P^2}+T_{A_1}\sigma_1E'q(1+x)$$

$$\left.+T_{A_1}\sigma_1Ep_1(1+x)+S_1x\sigma_1\left(\frac{1}{\in}+F\right)(1+x)+S_1x\sigma_1E\frac{p_1}{P}\right]$$

Eliminating R_1 between these, we get

$$\Rightarrow A_2C_1^2+A_1C_1+A_o=0 \tag{42}$$

where we have put $C_1=\sigma_1^2$ and $b=(1+x)$ and

$$A_2=b\left[\frac{1}{\in}+F\right]^2E'^2q^2\left[b\left(\frac{1}{\in}+F\right)+E\frac{p_1}{P}\right]$$

$A_1=$

$$\left(\frac{1}{\in}+F\right)^3b^4+\left[E\frac{p_1}{P}\left(\frac{1}{\in}+F\right)^2\right]b^3+\left[\frac{E'^2q^2}{P^2}\left(\frac{1}{\in}+F\right)\right]b^2+\left[E'^2q^2\left(\frac{Ep_1}{P^3}-T_{A_1}\left(\frac{1}{\in}+F\right)\right)+(b-1)S_1\left(\frac{1}{\in}+F\right)^2\right.$$

$$\left.(Ep_1-E'q)\right]b+\left[T_{A_1}E'^2q^2E\frac{p_1}{P}\right]$$

$$A_0=\frac{1}{P^2}\left(\frac{1}{\in}+F\right)b^4+\left[\frac{Ep_1}{P^3}-T_{A_1}\left(\frac{1}{\in}+F\right)\right]b^3+\left[T_{A_1}E\frac{p_1}{P}\right]b^2+\left[\frac{1}{P^2}(b-1)S_1(Ep_1-E'q)b\right] \tag{43}$$

Since σ_1 is real for overstability, both the values of $C_1 (= \sigma_1^2)$ are positive. Equation (42) is quadratic in C_1 and does not involve any of its roots to be positive if

$$Ep_1 > P^3 \; T_{A_1} \left(\frac{1}{\in} + F \right) \text{ and } Ep_1 > E'q, \tag{44}$$

which imply that

$$\kappa < \frac{\in^3 \; Ev^3 d^4}{4\Omega^2 \pi^4 k_1^2 (k_1 + v' \in)} \text{ and } E'\kappa < E\,\kappa' \tag{45}$$

Thus, $\kappa < \dfrac{\in^3 \; Ev^3 d^4}{4\Omega^2 \pi^4 k_1^2 (k_1 + v' \in)}$ and $E'\kappa < E\,\kappa'$ are the sufficient conditions for the non existence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

REFERENCES

REFERENCES

1. Bhatia, P.K. and Steiner, J.M	1973	J. Math. Anal. Appl. 41, 271.
2. Boussinesq, J.	1903	Theorie Analytique de la Chalecer, 2, 172, Gauthier-Villars, Paris.
3. Chandrasekhar, S.	1957	'Thermal Convection', Daedalus, 86,323.
4. Chandrasekhar, S.	1961	"Hydrodynamic and Hydromagnetic Stability," Clarendon Press, Oxford.
5. Chandrasekhar, S.	1981	"Hydrodynamic and Hydromagnetic Stability,"Dover Publication, New York.
6. Darcy, H.	1856	'Les Fountains Publiques de la ville de Dijon,' Dalmount, Paris.
7. Drazin, P.G. and Reid, W.H.	1981	'Hydrodynamic Stability,' Cambridge University Press, Cambridge.
8. Garg, A., Srivastava, R.K. and Singh,K.K	1994	Proc.Natn.Acad.Sci.(India) A64, 355.
9. Jeffreys, H.	1926	'The stability of layer of liquid heated from below', Phil.Mag. 2, 833
10. Jeffreys, H.	1931	Cartesian Tensors, Cambridge, England.
11 Joseph, D.D.	1976	"Stability of Fluid Motions II, Springer Verlag, New York.
12. Joshi, A.K.	1976	Acta Ciencia Indica 24,377
13. Lapwood, E.R.	1948	Proc.Camb.Philos.Soc.44,508.
14. Manegold, E.	1973 a,b	Kolloid-z., 80,253.

15. Manegold,E. 1941 Kolloid-z, 81, 19.

16. Oldroyd, J.G. 1958 Proc. Roy.Soc.London A 245, 278.

17. Rayleigh, Lord 1916 'On convective currents in a horizontal layer of fluid when the higher temperature is on the underside', Phil.Mag., 32, 529.

18. Rivlin, R.S.and Ericksen, J.L. 1955 J. Rational Mech.Anal. 4, 323.

19. Sharma, R.C. 1975 'Thermal instability in a viscoelastic fluid in hydromagnetics', Acta Physica Hungarica 38, 293.

20. Sharma, R.C. and Kumar, P. 1996 Z. Naturforsch, 51a, 821.

21. Srivastava, R.K. and Singh, K.K 1988 Bull .Cal. Math.Soc. 80, 286

22. Veronis,G. 1965 J.Marine Res.23,1

23. Vest, C.M. and Arpaci, V.S. 1969 'Overstability of a viscoelastic fluid layer heated form below', J.Fluid Mech., 36, 613.

24. Wooding, R.A. 1960 J.Fluid Mech. 9, 183.