A STUDY ON THE GRAVITATIONAL POTENTIAL DUE TO SOME NON-SPHERICAL BODIES AND APPLICATIONS

Dissertation Submitted in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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By

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Certificate

*We certify that the Thesis entitled "**A Study on the Gravitational Potential due to some Non-Spherical Bodies and applications**" submitted by **Smti. Rinku Chakravarty** for the degree of Doctor of Philosophy of Assam University, Silchar embodies the record of original investigation carried out by her under our supervision. She has been duly registered and the thesis presented is worthy of being considered for the award of the Ph.D. degree. This work has not been submitted for any degree of any other University.*

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Declaration

I, Rinku Chakravarty, a Ph.D. Scholar in Mathematics, Assam University, Silchar, do hereby solemnly declare that I have duly worked on my Ph.D. Thesis under the supervision of Prof. T.Som, Department of Mathematics, Prof. A.K. Sen, Department of Physics and Prof. D.Biswas, Department of Mathematics, Assam University and I have not submitted this work to any other degree.

I also declare that the title of my Thesis is

A STUDY ON THE GRAVITATIONAL POTENTIAL DUE TO SOME NON-SPHERICAL BODIES AND APPLICATIONS.

Rinku Chakravarty 4/02/08

Rinku Chakravarty

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Rinku Chakravarty
4/02/08

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Contents

Ph.D

RESEARCH PROPOSAL

1. Title: **A study on the Gravitational Potential due to some non-spherical bodies and applications.**

2. Statement of the problem: The Newton's law is one of the greatest discoveries which scientific investigation ever yielded to mankind. According to this law, every matter of particle attracts every other particle with a force which is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

The astronomical observations of the Danish astronomer Tycho Brahe on the motion of Mars led Kepler to formulate the law of planetary motion. Newton gave us an explanation of planetary motion in terms of the force of gravity which exists between the sun and the planet. Thus, Newton's law of gravitation hold even when r is large (except in the case, where relativistic corrections are appreciable). It should also be noted that Newton's law is valid even when r is very small but ceases to hold when r is of the order of 10^{-7} cm. In this case, instead of attraction, there is repulsion, because if there is attraction between two atoms separated by a distance of the order of 10^{-7} cm. the atoms will superimpose on each other which is practically impossible. Actually the force in this case is of electrical nature.

Thus we see that the law is applicable to all distances ranging from zero to infinity.

For example, the law of gravitation is true at even very large distances like the distance between galaxies. Gravity appears to exist at even distances of tens of millions of light years. In a galaxy we have a scale of 50,000 to 100,000 light years for measuring distances between heavenly bodies. The earth's distance from the sun is 8 1/2 light minutes, so we see how large these dimensions are. It should be noted that the law, unlike Inverse Square law; as in the case of electrostatics and magnetism, is not influenced by the medium in which the force acts. It is also independent of chemical composition of the attracting masses, and their temperatures. Thus we find that the law is universal.[(Chatterjee and Sengupta, 2001; Feynman et al., 1996; Gamow and Cleveland,1968; Narlikar,1993).

In a gravitational field, to move a unit mass from one point to another point requires an expenditure of work. This amount of work done may be negative or positive according as the body is moved in the direction or against the direction of the force of attraction. This amount of work done is a measure of the potential difference between the two points between which the unit mass is moved. The gravitational potential at a point in a gravitational field is measured by the amount of work done in bringing a unit mass from infinity to that point (Gupta 1997).

The study of Gravitational potential due to masses of different shapes is a very old problem, but it has a lot of importance in Astrophysics and other branches of Physics. Since the time of Newton, this problem was addressed by a number of scientists, and still there are a number of questions which remain unsolved.

For a spherical mass distribution the potential is generally given by the expression (between two mass points m1 and m2, separated by distance r)

$$P = -G\frac{m_1 m_2}{r}$$

One can also evaluate expressions if the gravitating body is a finite cylinder. However, the solution is not very straight forward and the expression is given in terms of some series, which converge only under certain conditions. This problem also has implications in general Theory of relativity. The calculation of gravitational potential due to ellipsoidal masses, finite disk and spirals are important in Astrophysics, as the galaxies are of these shapes. A knowledge of the gravitational potential due to such bodies will help to understand the dynamics of cluster of galaxies and gravitational bending of light due to galaxies of various shapes.

In this work we plan to carry out detailed calculations to find out gravitational potential due to such masses.

3. Survey of literature: Preliminary calculations on gravitational potential due to some non- spherical masses are available in literature (Chatterjee and Sen Gupta 2001). We shall also consider the motion of photon under the influence of gravitational potential of different masses. The motion of photon in a gravitational field can be taken as a motion in a dielectric medium (Schleich & Scully 1984). Thus the gravitational field may influence the state of polarization of the electromagnetic wave i.e. we expect a Faraday kind of rotation θ_F, due to gravitational field. But there is no Gravitational Faraday effect observed in case of sun. The result was derived by Plebanski (1967)'s equation for θ_F. There are various other works on gravitational effects such as Quantum Gravitational Diffusion and gravitational effects such as Quantum Gravitational Diffusion and Stochastic fluctuation in the velocity of light (Anderson et al. ,1975; Bergstrom and Goobar ,1999; Counselman et.al., 1974; Dyson et.al., 1920; Einstein, 1905; Einstein ,1975 ; Ellis et al., 2000) related to polarization and other phenomenon.

4. Objective: In this work, we shall calculate expression for potential due to different gravitational masses.

5. Data and Methodology: We will look into journals for different treatments of gravitational potential calculations. Such calculations have been also done in GTR to find expression for metric tensor(g).

6. Organization: The present work will be organised in the form of chapters comprising of Introduction, discussion on the methods of determining the gravitational potential of non-spherical masses. The thesis will include the following chapters:

In Chapter-1, we will discuss the discovery of the laws of gravitation and some of its consequences, its effects on the history and refinements of gravitational laws by Einstein.

In Chapter-2, we will discuss about gravitational potential, its physical meaning, equipotential surface, Laplace's and Poisson's equations for potential.

In Chapter-3, we will discuss the methods of finding gravitational potential at any point on the axis or on the plane of the non-spherical masses like disc or cylinder.

In Chapter-4, we will discuss the gravitational potential of a non-spherical mass like cylinder at any point, not required to be only on the plane or on the axis, outside the body.

In Chapter-5, we will calculate the gravitational potential of an ellipsoidal mass of Prolate shape, at any point, not necessarily to be on the axis, outside the body.

In Chapter-6, we will calculate the gravitational potential of an ellipsoidal mass of Oblate shape, at any point, not necessarily to be on the axis, outside the body

In Chapter-7, the summary of the work along with the important conclusions will be presented.

A Knowledge of the gravitational potential due to such bodies will help in understanding the dynamics of cluster of galaxies and gravitational bending of light due to galaxies of various shapes.

1

Introduction

Everybody is intrigued by the story of our origins viz., how the Universe came into being, how it got to be way it is, and why it is a suitable home for life forms like ourselves. The questions 'where do we come from?' , 'where do we live?' are the most profound questions and the ability to provide reasonably complete answers to those questions arguably rank as the greatest achievement of human thought.

Astronomy is usually considered as the oldest observational science and can also claim to be the youngest of the modern scientific disciplines when combined with the spectacular development of its sister discipline of theoretical Astrophysics (Basu 2001). This is apparently a contradictory situation and can be understood when one considers the fact that man's eternal quest to comprehend the splendour of the heavens had no beginning, and it will never cease. Astronomy and Astrophysics therefore stand as a symbol for man's ceaseless endeavour to know the unknown, to see the unseen and to understand the origin and evolution of the physical universe. The days of Aristotle are gone! He emphatically declared- "Man can never unfold the mysteries hidden within the stars"(Gribbin 1999). Now, it is possible to explore many of the mysteries of the universe by combining modern technological skill with the application of the laws of mathematics and modern physics.

Virtually, all of our information about the Universe at large comes from studies of electromagnetic radiation viz., light, radio waves, X-rays a and other variations on the theme- all of which travels at the speed of light, 30,000 million cm per second(Basu 2001, Gribbin 1999).

This is certainly a very large speed! However, at thc same time , the Universe is also very large, so that light and other forms of electromagnetic radiation take a long time to reach us from other stars and galaxies Even from a relatively nearby star, light spends years on its journey to Earth, so that we see the star as it was years ago, when the light left it. We detect light from galaxies and quasars so remote that the light has been millions, hundreds of millions or even, in some cases, thousands of millions of years on its journey across space to us, and we see these objects as they were that long ago, when the Universe was correspondingly younger. Here, we have talked about galaxies, in the following let us see what they are:

Galaxies are huge collections of stars, held together by gravity to form an 'island' in space. The largest galaxies contain thousands of billions of stars and may be served hundred thousand light years in diameter Even the small 'dwarf' galaxies contain millions of stars. **Milky Way** galaxy, in which we live, contains a few hundred billion stars. The size of a galaxy if compared with the size of the orbit of the Earth around the sun is about the same proportions as the size of our body compared with an individual atom (Basu 2001, Gribbin 1999,Hodge 1986).

Although, the galaxies are having very big sizes, most galaxies are so far away from us that they can be seen with the aid of telescopes Only, the nearest large galaxy, the **Andromeda** galaxy, and two small companions to the **Milky Way** , the **Magellanic** Clouds, can be seen as faint patches of light on the sky with the unaided human eye. About 50 billion galaxies are visible to modern telescopes (including the **Hubble** Space **Telescope**), but only a few thousand have been studied systematically(Basu 2001, Chown 1993, Harrison 1981, Hawking 1988).

CLASSIFICATION OF GALAXIES: (Barnes and Hernquist 1992, Basu 2001; Caldwell and Ostriker 1981; Mavridis 1971; Oegerle (Ed) , Fitchett and Danly 1990; Wailen 1990) :Galaxies are broadly divided into two classes viz., **elliptical** and **disc** galaxies, by their appearance. In addition to the visible bright stars we can see, galaxies are embedded in large amount of **dark matter,** revealed by its gravitational influence on the way galaxies move. Most of the galaxies occur in **clusters,** the most distant galaxy known (dubbed 8C 1435 + 635 and identified in 1994) has a **redshift** of 4.25, and is seen by light which left it when the Universe was only 20 per cent of its present age.

The galaxies were first classified by Hubble in 1920s into different types according to their shape and structural features. Later, Alam Sandage introduced some modifications in this classification and compiled a catalogue of them.

However, much earlier than the above workers, Charles Messier and J.L.E.Dreyer made an extensive observation of galaxies (know to them as nebulae) and globular clusters and made catalogues; the nomenclature of which are still in use. They represented the celestial objects by numbers in the catalogues. An object in the Messier Catalogue is abbreviated by the letter M followed by a number. For example, M31 represents **Andromeda** nebula. The Dreyer's catalogue which is a revised version of John Herschel's general catalogue of Nebulae is known as the New General Catalogue (NGC). Any individual object belonging to this catalogue is identified by a number. For example, NGC 598 and NGC 205 represent the spiral galaxy in the **Triangulam** and a companion of **Andromeda** spiral respectively.

The supplements of the NGC that include discoveries up to 1908 are named as the Index Catalogues or simply IC. The majority of the NGC and IC objects are the galaxies. The galaxies enlisted in the NGC catalogue are brighter than those in the IC catalogue.

There are several classification schemes for galaxies. However, the most popular one is that given by Hubble. His scheme consists of three regular classes viz., ellipticals, spirals and barred spirals. The irregular galaxies (Irr I and Irr II) form fourth class.

The elliptical galaxies are sub-classified by Hubble according to their flattening or elasticity. For example, the spherical galaxies are termed as E0 and the extremely flattened ellipticals as E7.The galaxies with intermediate ellipticities are designated as E1, E2, E3, ...,E6.

The Hubble's classification of elliptical galaxies was based on the shapes of the images and not on their true shapes.

An E7 galaxy is a very flat elliptical galaxy but it may be seen nearly edge-on, and an E0 galaxy may be of any degree of ellipticity but appears so as seen face-on. The numbers 0 to 7 represent the flattening of the galaxies is defined in terms of the major and minor axes a and b by the relation $10(1 - (b/a))$ where $b/a = 1$ for E0 and $b/a = 3/10$ for E7, the other values lying in between. The elliptical galaxies contain mostly Population II stars and consist of no spiral feature. These galaxies have much greater ranges in size, mass and luminosity as compared to the spirals.

A spiral galaxy comprises of a nucleus, a disk, a corona or halo and spiral arms. The spiral arms of a galaxy consist of the Population I stars whereas the nucleus and the corona contain mainly Population II stars. A large number of spiral galaxies have "bars" passing through their nuclei. In a normal spiral galaxy, the spiral arms originate from two diametrically opposite sides of the nucleus, whereas in the case of 'barred' spirals, they begin from the end of the bar. Examples of spiral galaxies are M31, M33 etc. and that of barred spiral is NGC 1300.

Hubble denoted the normal spirals by S and the barred spiral by SB. The sub-classes are represented by the letters a,b,c. Thus, Sa and SBa are spirals and barred spirals respectively at one extreme, having large luminous nucleus and arms tightly coiled. On the other end, Sc and SBc are those in the other extreme with small nuclei and loose extended spirals(c.f. Figure 1.1).

Some disk-shaped galaxies are observed with no trace of spiral arms. Hubble assumed these galaxies to be intermediate between spirals and elliptical and designated them as S0 galaxies.

Among those galaxies so far found in the northern sky about 3 per cent are classed as irregulars. They show no trace of circular or rotational symmetry. Rather they have chaotic appearances. They are sub-divided into two sub-classes. The first group that show high percentage of stars with some emission nebulae and are denoted as Irr I galaxies. The example of this type are the Large and Small Magellanic Clouds, and our nearest pair of galaxies. Star clusters, variables, supergiants and gaseous nebulae and also both Population I and II stars are found in them The second type of irregular galaxies designated as Irr II galaxies, are similar to the first type in regard to lack of symmetry but they show no resolution into stars or clusters. Examples of such galaxies are NGC 3034 (M82) and NGC 5195 (a companion to the spiral galaxy M51).

Partly because of our existence inside and partly because of the presence of dust acting as a type of fog lying along the plane of the **Milky Way**, it is not easy to have a clear idea of what our own Galaxy would look like. It would have been the best if we could see our galaxy from outside! But in recent years various painstaking and difficult researches have revealed that our galaxy, if we could see it from outside, would have the general appearance exactly similar to that of M31 in **Andromeda** which belongs to the Sb type of galaxies, intermediate between the two extreme sub-classes of the spiral type.

Since the galaxy is a gigantic system consisting of innumerable stars and clusters and also of gas, dust and magnetic field, we have to study various aspects of it in order to understand it clearly.

OUR GALAXY (Barnes 1992, Basu 2001, Gribbin 1999, Mavridis 1971): Our home in the cosmos, an island of stars (hundred-s of billions of stars broadly similar to our Sun), gas and dust held together by gravity to form a disc galaxy some 30 kiloparsecs across surrounded by a halo of visible globular clusters, and embedded in a much more extensive halo of dust matter, detected only by its gravitational influence. The most obvious visible feature of our Galaxy is the band of light on the sky called the **Milky Way**. The **Milky Way** proper is a faint band of light across the right sky (bright at the Southern Hemisphere than in the north), which is seen by the telescopes to be made up of a vast number of individual stars, too faint (and to close together on the sky) to be picked out as individuals by the unaided human eye. This band of light is actually our view of the billions of stars that make up the disc of our Galaxy, from a viewpoint embedded in the disc, about two-third of the way out from the centre of the galaxy to the edge of the disc.

In places, the **Milky Way** is obscured by dark clouds of interstellar matter, so that it looks as if there are holes in the **Milky Way**. Observations using radio telescopes have shown, however, that the material of the **Milky Way** is genuinely distributed in a disc, with spiral arms of material trailing through the disc. Our Galaxy is indeed a disc galaxy like the millions of others we see, with the aid of telescopes, scattered across the sky

Although it is 30 kiloparsecs across (with the sun about 9 kiloparsecs out from the centre), the disc of our Galaxy is only about 300 parsecs thick in its outer regions. Like other disc galaxies, at the centre of our galaxy there is a bulge of stars, resembling a small elliptical galaxy. This bulge is about 7 kiloparsecs across and 1 kiloparsec thick; it lies in the direction of the constellation **Sagittarius**, as seen from the Earth. There may be supermassive black hole at the very centre of the galaxy The central region of the galaxy like the halo, contains only the old stars of about 15 billion years old (i.e., Population

II stars), and very little gas or dust; the disc contains stars of all ages, including young Population I stars.

Population II stars are thought to be left over from the first burst of star formation when the galaxy itself formed The youngest stars are concentrated in a thin layer about 500 parsecs thick at the centre of the disc, where star formation is still going on. Slightly older stars (2 to 5 billion years old) are spread through the entire disc. The **Sun** is one of these intermediate-age stars.

The stars of the disc, along with the clouds of gas and dust, orbit around the centre of the galaxy in a similar way to the way in which the planets orbit around the **Sun**.

The speed of each star in its orbit around the centre of the galaxy depends on its distance form the centre- stars further out from the centre move more slowly than those close to the centre. The **Sun** is moving at about 250 km/sec in its orbit, and takes roughly 225 million years to travel once around the Galaxy (an interval sometimes referred to as the cosmic year) Studies of the way in which the stars move indicate the nature of the gravitational field of the Galaxy as a whole, and reveal its overall mass. This is about 1,000 billion times the mass of our Sun, roughly ten times the mass of all the stars in the **Milky Way** put together. This is strong evidence for the existence of dark matter in our Galaxy, extending far beyond the bright disc of stars in the **Milky Way**.

THE MASS OF THE GALAXY(Basu 2001, Gribbin 1999).: The determination of the mass of the galaxies is an important subject and has made many astronomers interested to that for last many decades. Prior to the construction of the rotation curve of the galaxies in early fifties, the astronomers had to rely on approximate models of distributions of mass and velocity of the material in the galaxy which finally could lead to its total mass. The determination of the mass of our galaxies are done by many workers viz., Camm (1938) obtained M= 1.77 X 10^{11} M⊙, Lohmann (1953, 1956) obtained in two models M= 2X10^{11} and 2.5X 10^{11} M⊙ and Bucerious (1934) obtained M= 2 4 X 10^{11} M⊙. Takase, Perek, Safronov etc. obtained the masses of the galaxies in the range 0.6-0.8X 10^{11} M⊙.

V C.Rubin and her co-workers indicated conclusively that galaxies are much more massive and are much greater extension, in general. The mass, however, exists in unobserved form. Our galaxy has a radius of about 50 kpc with a total mass of the order of 10^{11} M⊙.

GALAXY FORMATION AND EVOLUTION (Basu 2001; Gribbin 1999; Hawking 1988; Oegerle (Ed), Fitchett and Danly 1990, Mavridis 1971): Astronomers' ideas about the origins and evolution of galaxies changed dramatically in the year 1990s, thanks to the development of improve telescopes (including the Hubble Space Telescope) and instrumentation, which made it possible to look further back in time by studying fainter, more highly redshifted galaxies, seen as they were when the Universe was young. Until this breakthrough, it was widely accepted that the galaxies we see today all formed essentially at the same time, just after the Big Bang, and that they had each evolved internally as thc Universe aged. The new picture is of a dynamic, changing Universe in which galaxies compete with onc another for 'living space' , merge, and absorb other galaxies. The most dramatic difference betwcen the old and new pictures is that, whereas elliptical galaxics were once thought to be the oldest systems, they are now regarded as relatively recent products of interactions involving disc galaxies and other systems.

Disc galaxies (some times known as spirals) are much more common than ellipticals in photographs of the sky, but the biggest ellipticals are much bigger than any spiral. A typical disc galaxy, like our own Galaxy, may contain 100 billion stars, but the biggest ellipticals contain 100 times more stars. But there are also many dwarf ellipticals, some of which are no bigger than a globular cluster, containing about 1 million stars, Since many of these dwarf systems must be too faint to be seen, there must be many more ellipticals than we have yet identified

Astronomers used to think that ellipticals are old because they contain mainly cool, red stars, and very little dust and gas. Since cool, red stars are old, it was argued that elliptical galaxies are old. But although disc galaxies contain many hot, young stars, and star formation is still going on amidst the clouds of gas and dust in these galaxies, they do also contain many old (Population II) stars, concentrated in a central bulge and scattered through a spherical halo around the disc. The new evidence show that the old, red stars in elliptical galaxies have actually come from disc galaxies. Ellipticals form either when two spiral galaxies collide with one another stripping away their discs, or form mergers in which an existing elliptical galaxy swallows up a disc galaxy. This is why the biggest ellipticals are so big.

The evidence comes from observations of galaxies in the act of merging, and from computer simulations of such events. When two disc galaxies collide, the thin discs are stripped off and the systems merge into a single 'starpile' shaped like an elliptical galaxy. The stars themselves do not collide with one another, but the interacting gravitational fields of the two systems pull all the stars into one ball. Clouds of gas and dust from the colliding galaxies really do collide with one another, sending shock waves rippling through the new system and triggering a wave of star formation.

When a large elliptical galaxy absorbs a small disc galaxy, the elliptical grows and still looks like a single star system. But computer simulations show that many stars from the disc galaxy end up following similar orbits to one another inside the enlarged elliptical. Photographs show that this is indeed the case. Instead of being featureless starpiles, ellipticals are criss-crossed by stripes and arcs of brighter light, corresponding to the remains of disc galaxies that have been swallowed up but not yet fully digested.

The further out into the cosmos we look, the further back in time we see, because light takes a finite time to travel across space. In our part of the Universe, clusters of galaxies contain many ellipticals and have an overall redish colour. At distances corresponding to a look back time of about 5 billion years, the clusters are much bluer (showing that active star formation was more common then), and the Hubble Space Telescope has shown that many of the objects in these distant clusters are pairs of disc galaxies in the act of merging. Ellipticals today contain so little gas and dust only because it was all turned into stars during those mergers.

The largest ellipticals sit at the centers of clusters today, continuing to grow by absorbing any other galaxy that comes too close, like a spider getting fat by sitting in its web and waiting for food to come to its way. Only one per cent of all the galaxies we see today are actively involved in such mérgers, but they happen so quickly (compared with the age of a galaxy) that astronomers calculate that half of all the galaxies we now see have been involved in mergers with galaxies roughly the same size in the past 7 or 8 billion years. Mergers must have been even more common when the universe was younger and galaxies were closer together.

Disc galaxies themselves are now thought to have formed from mergers between smaller units early in the life of the Universe. The range of ages of the globular clusters, from about 14 billion years, suggests that our Galaxy formed over a period of several billion years from an amalgamation of about 1 million smaller gas clouds. When each 'new' gas cloud collided with the growing Galaxy, the shock wave would trigger a burst of star formation, forming a new globular cluster or making a contribution to the bulge of stars at the centre of the Galaxy. Left-over material would then settle down into the disc, eventually forming the spiral arms. Computer simulations show that this whole process of mergers is particularly effective in the context of dark matter models, with the gravity of the dark matter holding everything together. Indeed, if there were no dark matter in the Universe, galaxies as we know them probably could not have formed at all.

There are still puzzles that have yet to be explained concerning the origin and evolution of galaxies. For example, in the 1980s researchers at the AT & T Bell Laboratories identified huge numbers of dwarf galaxies at redshifts corresponding to a look back time of 2-3 billion years. We see these dwarf galaxies as they were when the earth was half its present age. At that time , life on Earth had not even emerged from the seas and on to the land; but if there had been astronomers alive on **Earth** then, their telescope would have shown the night sky blazing with blue dwarf galaxies, each about one-hundredth of the size of our Galaxy.

There were so many of these galaxies that their appearance on astronomical photographs today has been described as like 'cosmic wallpaper'. But we do not see anything like them active today.

It may be that those dwarf galaxies were an intermediate stage in the formation of disc galaxies, and have been absorbed into the spirals we see closer to home. Or it may be that the dwarf galaxies that made up the cosmic wallpaper simply burned out. Because they were so small, with relatively weak gravitational field, the supernovae resulting from the first wave of star formation in these tiny galaxies may have produced shock waves powerful enough to sweep all the remaining gas and dust in the dwarfs away into intergalactic space, leaving no raw material for new stars to form from. The modern equivalents of those dwarfs (or at least, their fossil remains) may still be there, but now containing only old, fading stars, too faint to be seen from Earth.

Before the era of the blue dwarfs, the Universe contained galaxies much larger in size than those we see today, but no larger in mass. They were simply more spread out because gravity had not had time since the Big Bang to pull all the pieces together into more compact form. Whole groups of galaxies evolved together like an ecosystem of living creatures, competing for raw materials (clouds of gas to be turned into new stars), absorbing one another, and adapting to the changing conditions as the Universe itself expanded and aged. Listening to astronomers talking about emerging populations, evolution and verifications that become extinct (like the blue dwarfs), it is sometimes hard to remember that they are indeed astronomers talking about galaxies, not biologists talking about the evolution of life on **Earth**.

To understand the dynamics of galaxies it is certainly important to know more and more about the galaxies. In this regard, the knowledge of Gravitational Potential due to bodies having mass and different shapes are necessary.

The study of Gravitational potential due to masses of different shapes is a very old problem, but it has a lot of importance in Astrophysics and other branches of physics. Since the time of Newton, this problem was addressed by a number of scientists and still there are a number of questions remain unsolved.

For a spherical mass distribution the potential is generally given by the expression (between two mass points, m_1 and m_2, separated by a distance r)

$$P = -Gm_1m_2 / r$$

One can also evaluate expressions if the gravitating body is a finite cylinder. However, the solution is not very straight forward and the expression is given in terms of some series, which converge only under certain conditions. This problem also has implications in General Theory of Relativity (Ellis, Mavromatos and Nanopoulos 2000). As the galaxies are having different shapes like elliptical, disc, spiral the calculation of Gravitational Potential due to ellipsoidal masses, finite disc and spirals are important in Astrophysics. Knowledge of the Gravitational Potential due to such bodies will help to understand the dynamics of cluster of galaxies and gravitational bending of light due to galaxies of various shapes.

However, before going into the details of the problem, in the following we briefly write the story of the discovery of the law of gravitation and discuss some of its consequences, its effect on history and refinements of the law by Einstein, pertinent to the present work.

1.1 **Planetary motions:** The story begins with the ancients observing the motions of planets among the stars, and finally deducing that they went around the **Sun**, a fact that was rediscovered later by Copernicus. Exactly how the planets went around the **Sun**, with exactly what motion, took a little more effort to discover. In the beginning of the 15^{th} century there were great debates as to whether they really went around the **Sun** or not. Tycho Brahe had an idea that was different from anything proposed by the ancients; his idea was that these debates about the nature of the motions of the planets would best be resolved if the actual positions of the planets in the sky were measured sufficiently accurately. If measurement showed exactly how the planets moved, then perhaps it would be possible to establish one or another viewpoint. This was a tremendous idea - that to find something out, it is better to perform some careful experiments than to carry on deep philosophical arguments. Pursuing this idea, Tycho Brahe studied the positions of the planets for many years in his observatory on the island of Hven, near Copenhagen.

He made voluminous tables, which were then studied by the mathematician Kepler, after Tycho's death. Kepler discovered from the data some very beautiful and remarkable, but simple laws regarding planetary motion [Feynman et al. (1996)].

1.2 **Kepler laws:** The three laws of planetary motion, discovered by Johannes Kepler early in the 17^{th} century, using data gathered by Tycho Brahe(Gribbin 1999). These laws are:

i. Each planet moves around the Sun in an ellipse, with the **Sun** at one of the foci.

ii. The radius vector from the **Sun** to the planet sweeps out equal areas in equal intervals of time.

iii. The squares of the periods of any two planets are proportional to the cubes of the semi major axes of their respective orbits: $T \sim a^{3/2}$.

1.3 **Developments in planetary dynamics:** While Kepler was discovering these laws, Galileo was studying the laws of motion. The problem was, what makes the planets go around? Galileo discovered a very remarkable fact about motion, which was essential for understanding these laws. That is the principle of inertia –if something is moving, with nothing touching it and completely undisturbed, it will go on forever, coasting at a uniform speed in a straight line.

Newton modified this idea, saying that the only way to change the motion of a body is to use force If the body speeds up, a force has been applied in the direction of motion. On the other hand, if its motion is changed to a new direction, a force has been applied side ways. Newton thus added the idea that a force is needed to change the speed or the direction of motion of a body. In fact, the law is that the acceleration produced by the force is inversely proportional to the mass, or the force is proportional to the mass times the acceleration. The more massive a thing is, the stronger the force required to produce a given acceleration. Moreover, because of the principle of inertia, the force needed to control the motion of a planet around the sun is not a force around the **Sun** but towards the **Sun** [Feynman et al. (1996)]

1.4 **Newton's Law of Gravitation:** From his better understanding of the theory of motion, Newton appreciated that the **Sun** could be the seat or organization of forces that govern the motion of the planets. Newton proved to himself that the very fact that equal areas are swept out in equal times is a precise sign post of the proposition that all deviations are precisely radian – that the law of areas is a direct consequence of the idea that all of the forces are directed exactly towards the **Sun**.

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Next, by analyzing Kepler's third law it is possible to show that the farther away the planet, the weaker the forces. If two planets at different distances from the **Sun** are compared, the analysis shows that the forces are inversely proportional to the squares of the respective distances. With the combination of the two laws, Newton concluded that there must be a force, inversely as the square of the distance, directly in a line between the two objects. Newton, further, supposed that this relationship applied more generally than just to the **Sun** holding the planets. It was already known, for example, that the planet **Jupiter** had moons going around it as the moon of the earth goes around the earth, and Newton felt certain that each planet hold its moons with a force. He already knew of the force holding us on the earth, so he proposed that this was a universal force – that everything pulls everything else.

Then the question arises – if the moon pulls the whole earth towards it, why doesn't the earth fall right 'up' to the moon? This can be understood by the balancing of forces. What balances? Because the earth does the same trick as the moon, it goes in a circle around a point which is inside the earth but not at its center. The moon does not just go around the earth, the earth and the moon both go around a central position, each falling towards this common position. This motion around the common center is what balances the fall of each.

The Newton's law of gravitation is one of the greatest discoveries which scientific investigation ever yielded to mankind.

According to this law, every particle of matter attracts every other particle with a force which is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

The astronomical observations of the Danish astronomer Tycho Brahe on the motion of **Mars** led Kepler to formulate the laws of planetary motion. Newton gave us an explanation of planetary motion in terms of the force of gravitation which exists between the **Sun** and the planet. Thus, Newton's law of gravitation holds even when r is large (except in the case, where relativistic corrections are appreciable) It should also be noted that Newton's law is valid even when r is very small but ceases to hold when r is the order of 10^{-7} cm. In this case, instead of attraction, there is repulsion, because if there is attraction between two atoms separated by a distance of the order of 10^{-7} cm, the atoms will superpose on each other which is practically impossible. Actually the force in this case is of electrical nature.

Thus we see that the law is applicable to all distances ranging from zero to infinity.

For example, the law of gravitation is true at even big distances like the distance between galaxies. Gravity appears to exist at even distances of tens of millions of light years. In a galaxy we have a scale of 50,000 to 100,000 light years. The earth's distance form the sun is 8½ light minutes, so we see how large these dimensions are. It should also be noted that the law, unlike Inverse Square law, as in the case of electrostatics and magnetism, is not influenced by the medium in which the force acts. It is also independent of chemical composition of the attracting masses, and their temperatures. Thus we conclude that the law is universal. [Chatterjee and Sengupta (2001); Feynman et al.(1996); Gamons and Cleveland (1968)]

1.5 Newton's Law of Gravitation and Einstein's Theory of Relativity

The Newtonian Mechanics is applicable to macroscopic bodies, which are traveling with velocity much less than that of light. When the velocity of a body is comparable to that of light, the Newtonian Mechanics fails. To eliminate this limitation of Newtonian Mechanics, Einstein in 1905 proposed the special theory of relativity which can explain the motion of a particle moving with a velocity(v); where v is ~ c as well as <<c. However, the special theory of relativity is applicable to inertial frames only and can not be applied to non-inertial or accelerated frame of reference. Einstein, in 1915, developed a theory of

relativity that explains both accelerated and non-accelerated motion. According to Einstein, the effects of a gravitational field is precisely the same as those due to uniformly accelerated frame of reference provided acceleration of the frame is equal and opposite to the acceleration that gravitational field would impart to a particle in its frame. This principle is known as Principle of Equivalence. The theory of gravitational field formulated on the basis of the theory of relativity is called the General Theory of Relativity. Taking into consideration the four dimensional space times and using the solution of vacuum Einstein's equation yield the motion of photon in a gravitational field. This is equivalent to the motion of the photon in medium with equivalent refractive index.

$$\mu = \frac{C_{grav}}{C} = 1 - \frac{2Gm}{C^2 r}$$

Thus we find that the gravitational field behaves as a refracting medium having refractive index different from 1. So, a photon in a gravitational field behaves as if photon in an optical medium having a refractive index given by the above expression.

It is mentioned above that the theory of relativity developed by Einstein (1915) explains both motion of accelerated and non-accelerated body based on the strong equivalence principle. Later he developed a mathematical theory relating gravitation with space-time interval (ds) between two points of an event $ds^2 = g_{ik}\, dx^i\, dx^k$ where g_{ik} is the metric tensor, whose values depend on the gravitational field. Einstein's General Theory of Relativity was able to explain the perihelion shift of mercury and the deflection of star light, which was not possible with the help of Newton's theory of motion and gravitation. The experimental verification of deflection of light by **Sun** was performed by Eddington et al (1920) during the total solar eclipse of 1919. The other various experiments performed to test the General Theory of Relativity were solar gravitational deflection of radio waves [Counselman et al (1974)], bending of microwave radiation in gravitational field of **Sun** [Fomalent and Sarmek (1975)], dual frequency measurement of the solar gravitational microwave deflection [Weiler et al 1975], time delay data from mariner 6 and 7 [Anderson et al (1975)], verification of signal retardation by solar gravity [Reasenberg et al (1979)], verification of the principle of equivalence for massive bodies [Shapiro et al (1976)], new test of the equivalence principle from lunar laser ranging [Williams et al (1976)].

In conclusion, we must say that Newton's law of gravitation is not discarded by Einstein's theory of relativity. It can be shown that Newton's theory of gravitation can be regarded as giving a first approximation to the results of Einstein's General Theory of Relativity. Actually Newton's theory holds good when the gravitational field is weak.

The primary objective of this work is to calculate the gravitational potential at any point outside of non-spherical solid bodies of shapes like cylinder, prolate and oblate. This study will help us in solving many problems in Astrophysics viz., galaxy dynamics, cluster dynamics, the deflection of light when light passes through the gravitational field due to non-spherical bodies like galaxies.

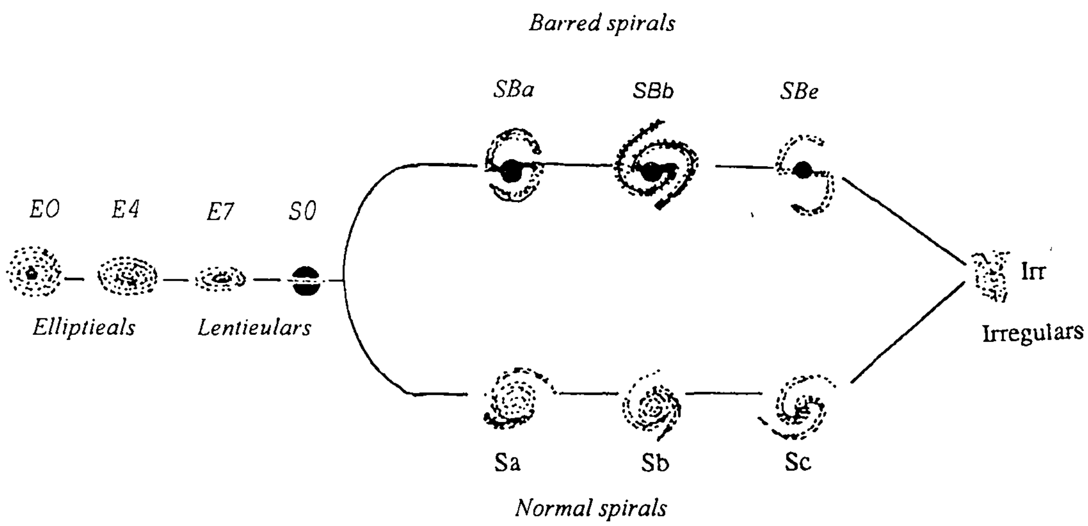

Figure 1.1 "Hubble's tuning fork diagram" showing four major classes of galaxies: 1.Ellipticals (E), 2. Lenticular (S0), 3. Spiral (S and SB) and 4. Irregulars (Irr)

2

Gravitational Potential

Gravity is the force of attraction that exists between any two objects that have mass. This is one of the four *fundamental interactions* known to physicists (Gribbin 1999).

Isaac Newton realized, in the 17^{th} century, that gravity acts in the same way for all objects anywhere in the Universe (that it follows a universal law), and that the attraction between two objects is proportional to the distance between them- the famous 'inverse square law of gravitation'(Gribbin 1999; Gamow and Cleveland 1968) .This law explains both the fall of an apple from a tree and the nature of the orbits of the Moon around the Earth and the planets around the Sun.

Albert Einstein explained the inverse square law, early in the 20th century, as a result of the way space time is distorted by the presence of matter(Gribbin 1999). His general theory of matter therefore goes further than Newton's theory of gravity, but includes Newton's theory within itself.

Gravity is the weakest of the four forces of nature, but because the gravitational influence adds up for every single particle in a lump of matter, and because the force has a very long range (in principle, infinite range), the overall effect of a lot of particles pulling together can be very strong(Gribbin 1999). The gravity of the Earth holds everything down on its surface, the gravity of the Sun holds planets in their orbits, and the gravity of everything in the galaxy holds the stars themselves in their orbits. In extreme cases, gravity can cause the collapse of space time into a black hole. Except for in the first split-second after the beginning of time (during the era of Inflation), gravity is the only force that has to be taken into account in describing the evolution of the Universe at large.

In this chapter we discuss various concepts and techniques used in determining Gravitational Potential (Chatterjee and Sengupta 2001; Gamow and Cleveland 1968; Gupta 1997), which are later utilized for the present work.

2.1 **Gravitational Potential**

In a gravitational field, to move a unit mass from one point A to another point B (c.f Figure 2.1) requires an expenditure of work. This amount of work done may be negative or positive according as the body is moved in the direction or against the direction of the force (f) of attraction. This amount of work done is a measure of the potential difference between the two points A (v) and B ($v+dv$) between which the unit mass is moved.

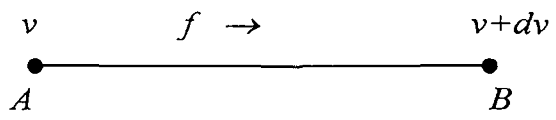

Figure 2.1

The gravitational potential at a point in a gravitational field is measured by the amount of work done in bringing a unit mass from infinity to that point

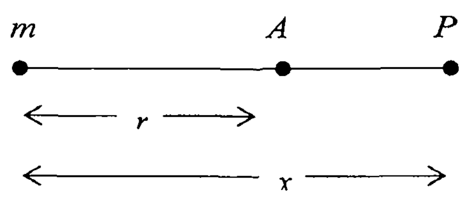

Figure 2.2

Now, the force of attraction on unit mass at P is Gm / x^2 and the work done in moving this unit mass through a distance dx in the direction of the force of attraction is $Gm / x^2 \, dx$.

Hence, the total work done moving the unit mass from infinity to the point A (c.f figure 2 2)

$$\int_{\infty}^{r} Gm/x^2 \, .dx = Gm / r$$

Thus, the potential at the point A is given by

$$V = - Gm / r$$

The negative sign indicates that the work is done on the particle since the particle is moved in the direction of attraction.

2 2 Gravitational Potential due to a system of attracting particles

Let us define a function by the relation

$$V = - \Sigma G \, m_s / r_s \quad \ldots\ldots \quad (2\ 1)$$

Where $r_s^2 = (a_s - x)^2 + (b_s - y)^2 + (c_s - z)^2 \quad \ldots\ldots\ldots\ldots (2.2)$

Thus, V is a function related to a system of attracting particles having a definite value at every point O' (Figure 2.3) of the space external

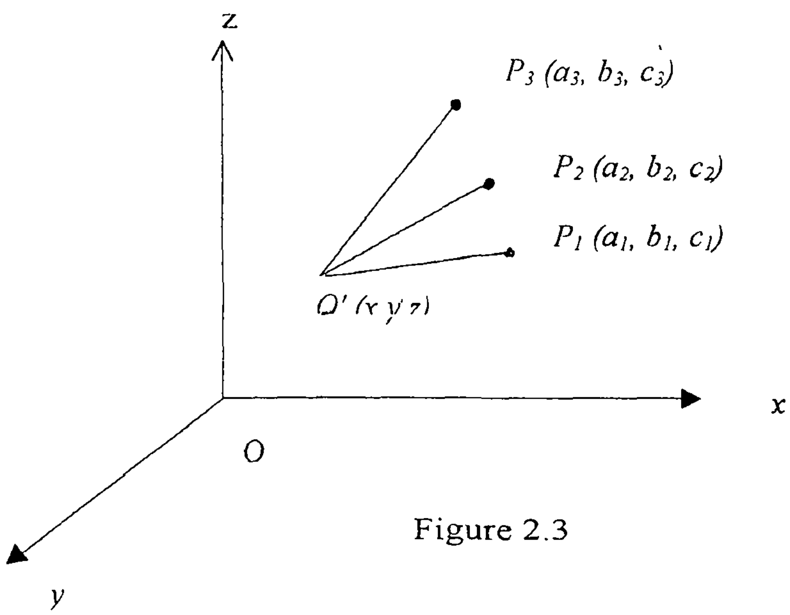

Figure 2.3

to the particles. It is a function of the co-ordinates (x,y,z) of the point O' and it is clearly a single valued function since it cannot have more than one value at each point O' . It does not depend on the particular system of axes of reference.

By differentiation of the expression (2.1), we have

$$\frac{\partial V}{\partial X} = +\sum \frac{Gm_a}{r_s^2}\frac{\partial r_s}{\partial x} \qquad (2.3)$$

Again from equation (2.3), by differentiation, we get

$$2r_s\frac{\partial r_s}{\partial x} = -2(a_s - x)$$

or $$\frac{\partial r_s}{\partial x} = -\frac{(a_s - x)}{r_s}$$

$$\therefore \frac{\partial v}{\partial x} = -\sum \frac{Gm_s (a_s - x)}{r_s^3} = -f_x$$

Similarly

$$\frac{\partial v}{\partial y} = -\sum \frac{G m_s (b_s - y)}{r_s^3} = -f_y$$

And

$$\frac{\partial v}{\partial z} = -\sum \frac{G m_s (b_s - z)}{r_s^3} = -f_z$$

The function V is called the potential of the system of attracting particles or the potential of the field of force. $-\frac{\partial v}{\partial x}, -\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial z}$ give the components of attraction in the directions of the axes. Since directions of the axes can be chosen arbitrarily, it follows that the space derivative of the potential V in any direction gives the component of attraction in that direction.

Let $\frac{\partial V}{\partial s}$ denote the derivative of potential in the direction ds

Therefore, we may write

$$\frac{dV}{ds} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial s}$$

$$= -(l f_x + m f_y + n f_z)$$

$$= -f$$

= field component in the direction ds,

where $l = \frac{\partial x}{\partial s}$, $m = \frac{\partial y}{\partial s}$; $n = \frac{\partial z}{\partial s}$ are the direction cosines of *ds*

Thus, we may define the potential at a point as a single-valued function of space; the derivative of which in a direction gives the intensity of the field in that direction.

In the vector notation, the above expression may be written as

$$\bar{f} = -\,grad\; V = -\,\bar{\nabla}\; V$$

2.3 **Physical Interpretation of Potential**

The total differential dV of the potential may be written in the form

$$dV = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy + \frac{\partial v}{\partial z}\,dz$$

$$= -(f_x\,dx + f_y\,dy + f_z\,dz)$$

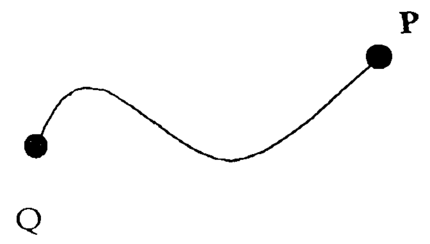

Figure 2.3

Hence, by integrating along the path from P to Q (c.f. Figure 2.3), we have

$$V_Q - V_p = -\int_P^Q \left(f_x \frac{\partial x}{\partial s} + \frac{\partial y}{\partial s} f_y + \frac{\partial z}{\partial s} f_z \right) ds \quad \text{.............(2.4)}$$

On the right hand side, the integral represents the work which the force of attraction would perform upon a particle of unit mass as if it moves along the path from P to Q, and the left-hand side which denotes the difference of potential between Q and P is therefore the work which the forces of attraction would perform upon a particle of unit mass as it moves along any path from P to Q. It is clear that the addition of a constant to the potential will not affect the values of the force components since the force components are obtained by differentiating the potential. Also by integrating the above expression, we get an expression for potential in terms of known force components together with a constant of integration.

This constant may be so chosen as to make the potential vanish at infinite distance from the attracting matter. Based on this consideration, we conclude that the potential at a given point due to a given attracting system is the work that would be done by the attractions of the system on a particle of unit mass as it moves along any path from an infinite distance up to the point considered.

2.4 Equipotential Surface

We know that the potential V of a given attracting system is a function of the coordinates *(x,y,z)*. Then the equation $V(x,y,z)$ = constant, represents a surface over which the potential remains constant. Such surfaces are called equipotential surfaces.

It follows from the definition of potential that only one such surface passes through any point of space so that no two equipotential surfaces can intersect. Also since V is constant over an equipotential surface, there is no difference of potential between any two points on this surface – as a result of which no work is done against the gravitational force in moving a unit mass along any path between the two points on this surface. Therefore, at every point on such a surface, the resultant attraction is normal to it through the point. That f (resultant attraction) is perpendicular to the equipotential surface may be shown in the following way:

Let $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point P(x,y,z) on the surface. Then $d\hat{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ lies in the tangent plane to the surface at P.

Now,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$=0[\because V(x,y,z)=\text{constant}] \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.5)$$

$$\text{or} \quad \left(\frac{\partial v}{\partial x}\hat{i}+\frac{\partial v}{\partial y}\hat{j}+\frac{\partial v}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2\ 6)$$

$$\text{or} \quad \vec{\nabla}V\cdot d\vec{r}=0 \quad \text{or} \quad \vec{f}\ d\vec{r}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.7)$$

i e $\vec{f}$ is perpendicular to $d\vec{r}$ and therefore to the surface

2 5 Laplace's Equation for Potential

Let V be the potential of a system of attracting particles at a point P(x,y,z) (c f Figure 2 4) not in contact with the particles, so that $V = -\sum Gm/r$, where m is the mass of the particle at (a,b,c) and r is the distance of P from the position of the mass m and is given by

$$r^2=(x-a)^2+(y-b)^2+(z-c)^2 \ . \qquad (2\ 8)$$

Therefore,

$$2r\,\frac{\partial r}{\partial x} = 2(x-a)$$

$$\text{or} \quad \frac{\partial r}{\partial x} = (x-a)/r \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.9)$$

Similarly

$$\frac{\partial r}{\partial y} = (y - \text{a})/ r$$

and $$\frac{\partial r}{\partial z} = (z - \text{a})/ r \quad . \quad ... \quad .. \quad .... \quad .... \quad .. \quad (2\ 10)$$

Now,

$V = -\Sigma Gm/r$

$$\therefore \frac{\partial V}{\partial x} = \sum \frac{Gm}{r^2} \cdot \frac{\partial r}{\partial x} = \sum \frac{Gm}{r^3} (r - \text{a}) \text{ (putting the value of } \frac{\partial r}{\partial x} \text{)} .. \quad (2.11)$$

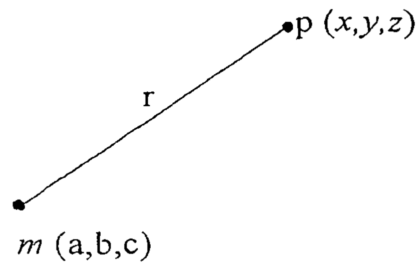

Figure 2.4

Similarly,

$$\frac{\partial V}{\partial y} = \sum \frac{Gm}{r^3} (y - \text{b})$$

$$\frac{\partial V}{\partial z} = \sum \frac{Gm}{r^3} (z - \text{c})$$

Further,

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left[\sum \frac{Gm}{r^3}(x-\text{a})\right] = \sum \frac{Gm}{r^3} - 3\sum \frac{Gm}{r^5}(x-\text{a})^2 \quad \ldots \quad (2.12)$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \sum \frac{Gm}{r^3} - 3\sum \frac{Gm}{r^5}(y-\text{b})^2 \quad \ldots \quad (2\ 13)$$

and

$$\frac{\partial^2 V}{\partial z^2} = \sum \frac{Gm}{r^3} - 3\sum \frac{Gm}{r^5}(z-\text{c})^2 \ldots\ldots \quad (2\ 14)$$

Hence, by addition, we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 3\sum \frac{Gm}{r^3} - 3\sum \frac{Gm}{r^5} \cdot r^2 = 0 \quad \ldots \quad (2\ 15)$$

or $\nabla^2 V = 0$; ∇ is called Laplacian operator,

or $\nabla, \nabla V = 0$

or div grad V = 0 (2 16)

This establishes Laplace's equation for potential Instead of considering isolated masses if we consider a continuous mass, we arrive at the same result

2.6 Poisson's Equation for Potential

Let the point P of coordinates (x,y,z) be inside of small radius R and centre (a,b,c) containing the point P. Since the sphere, we describe, is very small, we may regard the attracting mass within the sphere to be of uniform density ρ.

Now the matter which produces the potential V at P may be divided into two parts – the matter outside and the matter inside the small sphere.

Let V_1 denote the contribution towards potential at P by the matter outside the sphere, and V_2 the contribution towards potential at P by the matter inside the sphere. Since the point P is not in contact with the matter which produces the potential V_1, therefore, according to Laplace's equation $\nabla^2 V_1 = 0$ and V_2 being the potential at a point (x,y,z) inside a small sphere of radius R, we have

$$V_2 = -\frac{2}{3} G\pi\rho(3R^2 - r^2) \qquad (2.17)$$

Where r is the distance between the centre of the sphere, and the point P.

We have therefore,

$$\frac{\partial V_2'}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{2}{3}G\pi\rho\,(3R^2 - r^2)\right]$$

$$= \frac{2}{3}\pi\,\rho\,G\,2r\frac{\partial r}{\partial x}$$

$$= \frac{4}{3}\pi\,G\,\rho\,(x - a) \quad \left[\therefore \frac{\partial r}{\partial x} = (x - a)/\,r \right] \ldots (2.18)$$

$$\therefore \qquad \frac{\partial^2 V_2}{\partial x^2} = \frac{4\pi}{3}G\,\rho \quad \ldots\ldots (2.19)$$

Similarly, we have

$$\frac{\partial^2 V_2}{\partial y^2} = \frac{4\pi}{3}G\,\rho \quad \text{and} \quad \frac{\partial^2 V_2}{\partial z^2} = \frac{4\pi}{3}G\,\rho$$

or $\qquad \nabla^2 V_2 = 4\,\pi\,G\rho$ .... (2.20)

Now, V at every point at which there is attracting matter of density ρ, is given by

$$V = V_1 + V_2$$

$$\nabla^2 V = \nabla^2 V_1 + \nabla^2 V_2$$

$$= 0 + 4\pi\,G\rho$$

$$= 4\pi\,G\rho \quad \ldots\ldots (2.21)$$

This result is known as Poisson's equation for Potential.

3

Calculation of Gravitational Potential of some Non-Spherical Masses

In this chapter we discuss the method of calculation of Gravitational Potential of some uniform and regular non-spherical shaped bodies like disc and solid cylinder.

3.1 Calculation of Gravitational Potential of uniform Circular Disc of radius a and thickness t, at a point P on the axis

(Chatterjee and Sengupta 2001; Gamaw and Cleveland 1968):

The disc is considered to be divided into a number of concentric annuli or rings, and one such annulus has radius x and width dx (c.f. Figure 3.1)

The mass of this annulus is $2\pi x dx \rho t$, where ρ is the density of the material of the disc. Therefore, the potential at P due to the annulus under consideration is given by

$$dV = -G\frac{2\pi x dx \rho t}{\sqrt{r^2 + x^2}}$$

Thus the potential at P due to the whole disc is given by

$$V = -G2\pi\rho t\int_0^a \frac{xdx}{\sqrt{r^2 + x^2}} = -G\pi\rho t\int_0^a \frac{d\left(r^2 + x^2\right)}{\sqrt{r^2 + x^2}} = -2G\pi\rho t\left[\left(r^2 + x^2\right)^{\frac{1}{2}}\right]_0^a$$

$$= 2G\pi\rho t\left[r - \left(r^2 + a^2\right)^{\frac{1}{2}}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.1)$$

$$or \quad V = \frac{2GM}{a^2}\left[r - \left(r^2 + a^2\right)^{\frac{1}{2}}\right]$$

$$where\ M = mass\ of\ the\ disc = \pi a^2 \rho t$$

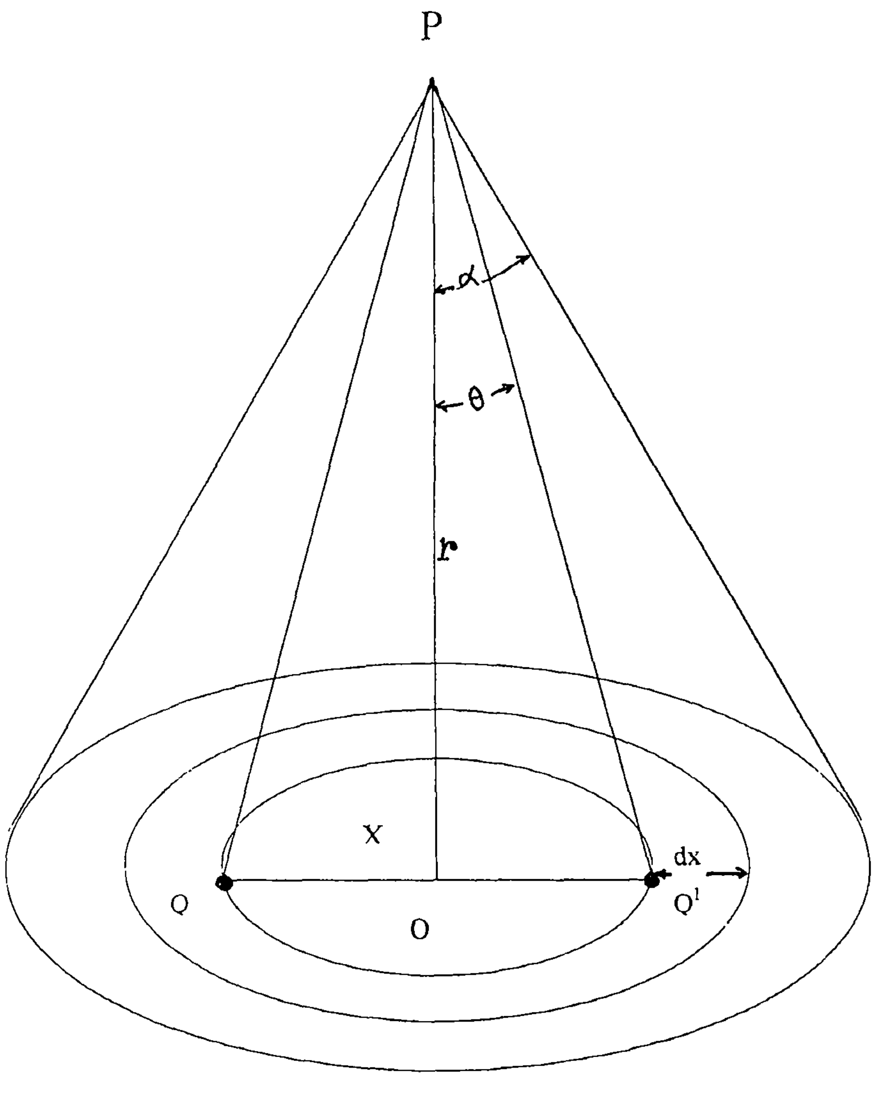

Figure 3.1

3.2 Calculation of Gravitational Potential of Homogeneous solid cylinder at a point on the axis of the cylinder (Chatterjee and Sengupta 2001; Gamaw and Cleveland 1968):

Here we discuss the method of calculation of Gravitational Potential of a homogeneous solid cylinder at a point P on the axis of the cylinder.

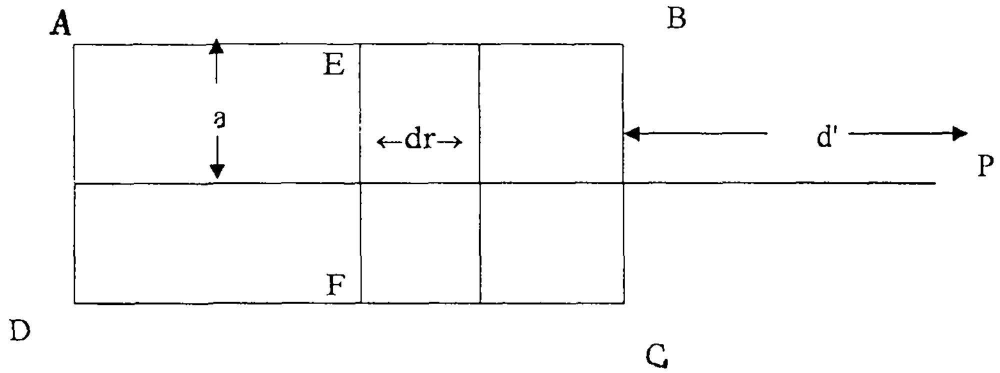

Figure 3.2

ρ is the density of the cylinder and a its radius (c.f. Figure 3.2). The cylinder is considered to be divided into a number of co-axial discs. One such disc EF of thickness dr lies at a distance r from the point P at which the potential is being considered. The potential at P due to the disc EF is given by

$$dV = 2\pi G\rho\left[r - \sqrt{r^2 + a^2}\right]dr \quad \text{..................(3.2)}$$

(from the expression for potential due to a disc c.f section 3.1)

Therefore, the potential at P due to the entire cylinder is given by

$$V = 2\pi G\rho \int_{r=d'}^{r=d'+l} \left(r - \sqrt{r^2 + a^2}\right) dr$$

$$= 2\pi G\rho \left[\left(\frac{r^2}{2}\right)_{d'}^{d'+l} - \int_{d'}^{d'+l} \sqrt{r^2 + a^2}\, dr \right]$$

Integration of the second term many be done in the following way:

Let $I = \int \sqrt{r^2 + a^2}\, dr = r\sqrt{r^2 + a^2} - \int \frac{r^2 dr}{\sqrt{r^2 + a^2}}$ ....(3.3)

[Integrating by parts]

Again

$$I = \int \frac{r^2 + a^2}{\sqrt{r^2 + a^2}} dr = \int \frac{r^2 dr}{\sqrt{r^2 + a^2}} + a^2 \int \frac{dr}{\sqrt{r^2 + a^2}} \quad \ldots\ldots\ldots\ldots\ldots(3.4)$$

From (3.3) and (3.4) we have

$$I = \frac{1}{2}\left[r\sqrt{r^2 + a^2} + a^2 \int \frac{dr}{\sqrt{r^2 + a^2}} \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5)$$

To evaluate the second integral of the above expression, let us put

$$r = a\tan\theta \quad or, \quad dr = a\sec^2\theta\, d\theta$$

$$\therefore\ a^2\int\frac{dr}{\sqrt{r^2+a^2}} = a^2\log\frac{r+\sqrt{r^2+a^2}}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.6)$$

When

$$I=\frac{1}{2}\left[r\sqrt{r^2+a^2}+a^2\log\frac{r+\sqrt{r^2+a^2}}{a}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.7)$$

$$\therefore\ V=2\pi G\rho\left[\frac{r^2}{2}-\frac{1}{2}r\sqrt{r^2+a^2}-\frac{a^2}{2}\log\frac{r+\sqrt{r^2+a^2}}{a}\right]_{d'}^{d'+l}$$

$$=\pi G\rho\left[l(l+2d')-(d'+l)\sqrt{(d'+l)^2+a^2}+d'\sqrt{d'^2+a^2}\right.$$

$$\left.-a^2\log\frac{(d'+l)+\sqrt{(d'+l)^2+a^2}}{a}+a^2\log\frac{d'+\sqrt{d'^2+a^2}}{a}\right]$$

$$=\pi G\rho\left[l(l+2d')-(d'+l)PA+d'.PB-a^2\log\frac{(d'+l)+PA}{d'+PB}\right]$$

$$\left[as\ (d'+l)^2+a^2=PA^2 \quad and \quad d'^2+a^2=PB^2\right]$$

$$= \frac{GM}{la^2}\left[l(l+2d') - (d'+l)PA + d'.PB - a^2 \log\frac{(d'+l)+PA}{d'+PB} \right]$$

Where, $M = \pi a^2 l \rho = mass\ of\ the\ cylinder.$

3.3 Calculation of Gravitational Potential of thin Uniform straight rod at any point(Chatterje and Sengupta 2001; Gamaw and Cleveland 1968):

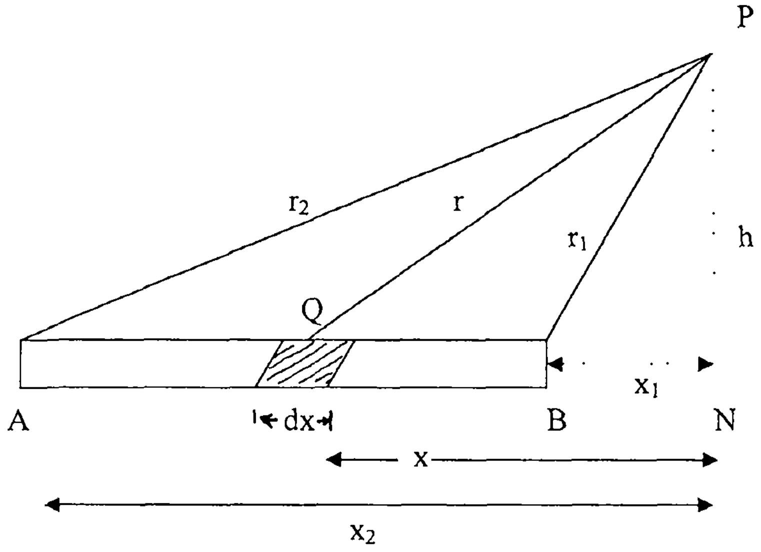

Figure 3.3

Point P, at which the potential due to the rod AB is to be considered (figure 3 3) m is the mass of the rod. Then mass per unit length of the rod is m/l, where l is the length of the rod

An element dx of the rod at Q is considered the mass of which is $m/l\ dx$

Hence, the potential at P due to this element is

$$dV = -\frac{G\frac{m}{l}dx}{r} = \frac{Gmdx}{l\sqrt{h^2 + x^2}} \qquad \ldots \qquad (3\ 8)$$

Therefore, the resultant potential due to the entire rod is given by

$$V = -\frac{Gm}{l}\int_{x=x_1}^{x=x_2} \frac{dx}{\sqrt{h^2 + x^2}} \qquad (3\ 9)$$

where,
$NQ = x$
$NB = x_1$
$NA = x_2$
$PN = perpendicular\ on\ AB\ produced = h$

Considering $x = h\tan\theta$

$\therefore dx = h\sec^2\theta\, d\theta$, the potential V is written as

$$V = -\frac{Gm}{l}\int \frac{h\sec^2\theta\, d\theta}{h\sec\theta} = -\frac{Gm}{l}\int \sec\theta\, d\theta = -\frac{Gm}{l}\left[\log(\sec\theta + \tan\theta)\right]$$

$$= -\frac{Gm}{l}\left[\log\frac{\sqrt{h^2+x^2}+x}{h}\right]_{x=x_1}^{x=x_2}$$

$$\left[\quad \log(\sec\theta+\tan\theta)=\log\left(\sqrt{1+\frac{x^2}{h^2}}+\frac{x}{h}\right)=\log\frac{\sqrt{h^2+x^2}+x}{h}\right]$$

$$= -\frac{Gm}{l}\left[\log\frac{\sqrt{h^2+x_2^2}+x_2}{\sqrt{h^2+x_1^2}+x_1}\right] = -\frac{Gm}{l}\left[\log\frac{r_2+x_2}{r_1+x_1}\right]$$

$where, r_1 = PB \quad and \quad r_2 = PA.$

Now from figure 3.3

$$r_2^2 - x_2^2 = h^2 = r_1^2 - x_1^2$$

or $$\frac{r_2+x_2}{r_1+x_1} = \frac{r_1-x_1}{r_2-x_2} = \frac{r_2+x_2+r_1-x_1}{r_1+x_1+r_2-x_2} = \frac{r_1+r_2+l}{r_1+r_2-l}$$

$$\therefore \quad V = -\frac{Gm}{l}\log\frac{r_1+r_2+l}{r_1+r_2-l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.10)$$

If we draw an ellipse with A and B as foci then for all points on the ellipse the sum of the focal distances will be the same. Here for the point P on the surface of the ellipse the sum of the focal distances is $(r_1 + r_2)$. Thus for all points over the surface of this ellipse the potential is constant. The potential is also constant over the ellipsoid obtained by revolving this ellipse about the line joining the foci, i.e , about AB.

It may be noted here that all the calculations on Gravitational Potential at points outside non-spherical masses like disc or cylinder, available in existing literature, are done considering the points either on the axis or on the plane of the disc / cylinder. However, in the following chapters, we calculate the Gravitational Potential of uniform solid cylinder, prolate and oblate shaped bodies at any point outside the body.

4

Calculation of Gravitational Potential of a Uniform Solid Cylinder at Any Point Outside The Cylinder

Calculation of the gravitational potential of a uniform solid cylinder at any point outside the cylinder is done in this chapter.

Let us consider a cylinder of length l (c.f. figure 4.1). The gravitational potential at a point $P(x',y',z')$ is to be calculated At first we calculate the gravitational potential due to the elementary disc at a distance z from the centre of the cylinder and then it will be integrated for the whole length of the cylinder for getting the total potential due to the whole cylinder.

We take spherical polar co-ordinate system $P(x',y',z')$; the point where potential is to be calculated

$$P(x',y',z') = P(RCos\theta' Sin\varphi', RSin\theta' Sin\varphi', RCos\varphi')$$

$$Q(x',y',z') = Q(rCos\theta, rSin\theta, z) \quad on\ disc$$

$$PQ^2 = (RCos\theta' Sin\varphi' - rCos\theta)^2 + (RSin\theta' Sin\varphi' - rSin\theta)^2 + (RCos\varphi' - z)^2$$

$$r'^2 = R^2 Sin^2\varphi'(Cos^2\theta' + Sin^2\theta') + R^2 Cos^2\varphi' + r^2(Cos^2\theta' + Sin^2\theta')$$
$$+ z^2 - 2RSin\varphi'(Cos\theta' Cos\theta - Sin\theta' Sin\theta) - 2Rz\,Cos\varphi'$$

$$= R^2 Sin^2\varphi' + R^2 Cos^2\varphi' + r^2 + z^2 - 2R\,Sin\varphi' Cos(\theta - \theta') - 2Rz\,Cos\varphi'$$

$$= R^2 + r^2 + z^2 - 2Rz\,Cos\varphi' - 2RSin\varphi' Cos(\theta - \theta')$$

$$= R^2 + r^2 + z^2 + H\,Cos\alpha + G_1 z$$

Where, $H = -2R\,Sin\varphi'$

$\alpha = \theta - \theta'$

$G_1 = -2R\,Cos\varphi'$

and $d\alpha = d\theta$

The Gravitational Potential at P is

$-d\psi = G\rho\, dr\, r\, d\theta\, dz / r'$

$$-d\psi = \frac{G\rho r dr d\theta dz}{\sqrt{R^2 + r^2 + z^2 + H\,Cos\alpha + G_1 z}} \qquad (\rho \text{ density per unit volume})$$

$$\therefore -\psi = G\rho \int_0^{R_0} \int_{-\theta'}^{2\pi-\theta'} \int_{-l/2}^{+l/2} r dr\, d\theta\, dz / \sqrt{R^2 + r^2 + z^2 + H\,Cos\alpha + G_1 z} \qquad \dots (4.1)$$

Where r: $0 \to R_o$ (radius of the cylinder)

$\theta: 0 \to 2\pi$

$\alpha: -\theta' \to 2\pi - \theta'$

and $z: -l/2 \to l/2$

Let $A = R^2 + z^2 + H\,Cos\alpha + G_1 z$

Then (4.1) changes to

$$-\psi = G\rho \int_{-\theta'}^{2\pi-\theta'} \int_{-l/2}^{+l/2} \left(\int_0^{R_0} r dr / \sqrt{A + r^2}\right) d\alpha\, dz \qquad \dots\dots\dots (4.2)$$

Now

$$\int_0^{R_0} r dr / \sqrt{A + r^2} \qquad \textit{puting } y^2 = A + r^2, \textit{ we get } 2y\,dy = 2r\,dr$$

$$= \int_{\sqrt{A}}^{\sqrt{A+R_0^2}} dy \qquad \textit{as } r \to 0, y \to \sqrt{A}$$

$$\textit{and } r \to R_o, y \to \sqrt{A + R_o^2}$$

$$= \sqrt{A + R_0^2} - \sqrt{A}$$

$$= \sqrt{R^2 + z^2 + H\,Cos\alpha + G_1 z + R_0^2} - \sqrt{R^2 + z^2 + H\,Cos\alpha + G_1 z}$$

$\therefore$ (4.2) Changes to

$$-\psi = \int_{-\theta'}^{2\pi-\theta'} \left[\int_{-l/2}^{+l/2} \left(\sqrt{R^2 + z^2 + H\, Cos\alpha + G_1 z + R_0^2} - \sqrt{R^2 + z^2 + H\, Cos\alpha + G_1 z} \right) dz \right] d\alpha$$

.... (4.3)

Let $B = R^2 + R_o^2 + H\, Cos\alpha$

$B' = R^2 + H\, Cos\alpha$

$$I_1 = \int_{-l/2}^{+l/2} \sqrt{R^2 + z^2 + H\, Cos\alpha + G_1 z + R_0^2}\, dz = \int_{-l/2}^{+l/2} \sqrt{z^2 + G_1 z + B}\, dz$$

$$I_2 = \int_{-l/2}^{+l/2} \sqrt{R^2 + z^2 + H\, Cos\alpha + G_1 z}\, dz = \int_{-l/2}^{+l/2} \sqrt{z^2 + G_1 z + B'}\, dz$$

$$\therefore -\psi = \int_{-\theta'}^{2\pi-\theta'} (I_1 - I_2) d\alpha \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4.4)$$

Calculation of $\int_{-l/2}^{+l/2} \sqrt{z^2 + G_1 z + B}\quad dz$

Let $z^2 + G_1 z + B = u^2 + \beta$

where $u = z + \frac{G_1}{2}$ *and* $\beta = B - \frac{G_1^2}{4}$

$\therefore \quad du = dz$

as $\quad z \to -\frac{l}{2}, \qquad u \to \frac{G_1}{2} - \frac{l}{2}$

$$z \to \frac{l}{2}, \qquad u \to \frac{G_1}{2} + \frac{l}{2}$$

$$I_1 = \int_{-l/2}^{+l/2} \sqrt{z^2 + G_1 z + B} \quad dz = \int_{G_1 - \frac{l}{2}}^{G_1 + \frac{l}{2}} \sqrt{u^2 + \beta} \quad du$$

$$= \frac{1}{2}\left[u\sqrt{u^2 + \beta} + \log(u + \sqrt{u^2 + \beta}) \right]_{G_1 - \frac{l}{2}}^{G_1 + \frac{l}{2}}$$

$$= \quad \frac{1}{2}\left[\frac{G_1 + l}{2}\sqrt{\left(\frac{G_1 + l}{2}\right)^2 + \beta} \; - \frac{G_1 - l}{2}\sqrt{\left(\frac{G_1 - l}{2}\right)^2 + \beta} \right] +$$

$$\frac{1}{2}\left[\ln\left\{ \frac{G_1 + l}{2} + \sqrt{\left(\frac{G_1 + l}{2}\right)^2 + \beta} \right\} - \ln\left\{ \frac{G_1 - l}{2} + \sqrt{\left(\frac{G_1 - l}{2}\right)^2 + \beta} \right\} \right]$$

$$= \frac{1}{8}\left\{ (G_1 + l)\sqrt{(G_1 + l)^2 + 4\beta} \; - (G_1 - l)\sqrt{(G_1 - l)^2 + 4\beta} \right\} +$$

$$\frac{1}{2}\left\{\ln\left(G_1 + l + \sqrt{(G_1 + l)^2 + 4\beta}\right) - \ln\left(G_1 - l + \sqrt{(G_1 - l)^2 + 4\beta}\right)\right\}$$

. ...(4 5)

Where $G_1 = -2RCos\varphi'$

$$\beta = B - \frac{G_1^2}{4}$$

$$= R^2 + R_0^2 + HCos\alpha - R^2Cos^2\varphi'$$

$$= R^2Sin^2\varphi' + R_0^2 + HCos\alpha$$

$$= \left(R_0^2 + R^2Sin^2\varphi'\right) + HCos\alpha$$

$$= D + HCos\alpha$$

Where, $D = R_0^2 + R^2Sin^2\varphi'$

$$\sqrt{4\beta + (G_1 + l)^2} = \sqrt{4(D + HCos\alpha) + (G_1 + l)^2}$$

$$= \sqrt{4D + (G_1 + l)^2 + 4HCos\alpha}$$

$$= \sqrt{4H}\ \sqrt{\frac{4D + (G_1 + l)^2}{4H} + Cos\alpha}$$

$$= 2\sqrt{H}\,\sqrt{K + Cos\alpha}$$

Where, $K = \dfrac{4D + (G_1 + l)^2}{4H}$

$$\int_{-\theta'}^{2\pi-\theta'} \sqrt{4\beta + (G_1 + l)^2}\, d\alpha = \int_{-\theta'}^{2\pi-\theta'} 2\sqrt{H}\,\sqrt{K + Cos\alpha}\, d\alpha$$

$$= 2\sqrt{H} \int_{-\theta'}^{2\pi-\theta'} \sqrt{K + Cos\alpha}\, d\alpha$$

$$= 4\sqrt{H}\,\sqrt{K + l}\; E(\pi, q_1) \ldots\ldots\ldots\ldots\ldots \text{(4.6)}$$

Calculation of $\int_{-\theta'}^{2\pi-\theta'} \sqrt{K + Cos\alpha}\, d\alpha$

Now from the integral

$$\int \sqrt{a+bCosx}\,dx = \int \sqrt{a+b\left(1-2Sin^2\frac{x}{2}\right)}\,dx$$

$$= \int \sqrt{a+b-2bSin^2\frac{x}{2}}\,dx$$

$$= \sqrt{a+b}\int \sqrt{1-\left(\frac{2b}{a+b}\right)Sin^2\frac{x}{2}}\,dx \qquad \cdots\cdots\cdots \text{(I) (say)}$$

We get,

$$\therefore \quad \int_{-\theta'}^{2\pi-\theta'} \sqrt{K+Cos\alpha}\,d\alpha, \ \textit{comparing with equation}\,\text{(I)}$$

$$= \sqrt{K+1}\int_{-\theta'}^{2\pi-\theta'} \sqrt{1-q^2Sin^2\frac{\alpha}{2}}\,d\alpha \qquad K=a, b=1,\ q_1^2=\frac{2}{K+1}$$

Let $\frac{\alpha}{2}=\gamma$

$d\alpha = 2d\gamma$

and let us take the case where x is in direction of P then $\theta'=0$, $\alpha=\theta'$ and as $\alpha \to 0$ to 2π, $\gamma \to 0$ to π, then

$$\int_{-\theta'}^{2\pi-\theta'} \sqrt{1-q^2 Sin^2 \frac{\alpha}{2}}\, d\alpha$$

$$= 2\int_0^{\pi} \sqrt{1-q^2 Sin^2\gamma}\ d\gamma$$

$= 2E(\pi, q_1)$ (Gradshteyn and Ryzhik, 2001)

Where the elliptic integral

$$E(\pi, q) = \int_0^{\pi} \sqrt{1-q^2 Sin^2\gamma}\, d\gamma$$

$$= \frac{2}{\pi} E\pi + Sin\pi Cos\pi \left(b_0 + \frac{2}{3} b_1 Sin^2\pi + \ldots\ldots\ldots \right)$$

$$= 2E$$

$$= \pi \left\{ 1 - \frac{1^2}{2^2} q^2 - \frac{1^2 3}{2^2 4} q^4 - \ldots\ldots\ldots\ldots\ldots\ldots - \frac{(2n-1!!)}{2^n n} \frac{q^{2n}}{2n-1} \right\}$$

Therefore, $\int_{\theta'}^{2\pi-\theta'} \sqrt{K+Cos\alpha}\, d\alpha = \sqrt{K+1}\, 2E(\pi, q_1)$

and $$\int_{-\theta'}^{2\pi-\theta'} \ln\left\{(G_1+l)+\sqrt{(G_1+l)^2+4\beta}\right\}d\alpha$$

$$= \int_0^{2\pi} \ln\left(G_1+l+2\sqrt{H}\sqrt{K+Cos\alpha}\right)d\alpha$$

$$= \int_0^{2\pi} \ln 2\sqrt{H}\left(\frac{G_1+l}{2\sqrt{H}}+\sqrt{K+Cos\alpha}\right)d\alpha$$

$$= \int_0^{2\pi} \ln 2\sqrt{H}\,d\alpha + \int_0^{2\pi} \ln\left(Q+\sqrt{K+Cos\alpha}\right)d\alpha$$

where $Q = \dfrac{G_1+l}{2\sqrt{H}}$

$$= 2\pi \ln 2\sqrt{H} + \int_0^{2\pi} \ln Q\left(1+\frac{\sqrt{K+Cos\alpha}}{Q}\right)d\alpha$$

$$\simeq 2\pi \ln 2\sqrt{H} + 2\pi \ln\frac{G_1+l}{2\sqrt{H}} + \int_0^{2\pi} \frac{\sqrt{K+Cos\alpha}}{Q}d\alpha \;......$$

(neglecting higher powers)

Using $\ln(1+x) = x - \frac{x^2}{2} + ...|x| \leq 1$

$$= \ln(G_1 + l) + \frac{1}{Q}\int_0^{2\pi} \sqrt{K + Cos\alpha}\, d\alpha|$$

$$= 2\pi \ln(G_1 + l) + \frac{2\sqrt{H}}{G_1 + l} 2\sqrt{K+1}\, E(\pi, q_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.7)$$

Similarly,

$$\int_{-\theta'}^{2\pi-\theta'} \sqrt{(G_1 - l)^2 + 4\beta}\ d\alpha \quad = 2\sqrt{H} \int_0^{2\pi} \sqrt{K' + Cos\alpha}\ d\alpha$$

$$\left(where \quad K' = \frac{4D + (G_1 - l)^2}{4H} \right)$$

$$= 2\sqrt{H}\ 2\sqrt{K'+1}\ E\left(\pi, q_1'\right) \ldots\ldots\ldots\ldots (4.8)$$

and

$$\int_{-\theta'}^{2\pi-\theta'} \ln\ \{(G_1 - l) + \sqrt{(G_1 - l)^2 + 4\beta}\ \}\, d\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.9)$$

$$= 2\pi \ln (G_1 - l) + \frac{2\sqrt{H}}{G_1 - l} 2\sqrt{K'+1} \quad E(\pi, q_1{}^1)$$

$$\therefore \int_{-\theta'}^{2\pi-\theta'} I_1 \, d\alpha = \frac{1}{8}(G_1 + l) \int_0^{2\pi} \sqrt{(G_1 + l)^2 + 4\beta} \, d\alpha - \int_0^{2\pi} \frac{1}{8}(G_1 - l)\sqrt{(G_1 - l)^2 + 4\beta} \, d\alpha$$

$$+ \frac{1}{2} \int_0^{2\pi} \ln(G_1 + l) + \sqrt{(G_1 + l)^2 + 4\beta} \, d\alpha - ½ \int_0^{2\pi} \ln(G_1 - l) + \sqrt{(G_1 - l)^2 + 4\beta} \, d\alpha$$

$$= ⅛ (G_1 + l) \int_0^{2\pi} \sqrt{(G_1 + l)^2 + 4\beta} \, da \quad - ⅛ (G_1 - l) \int_0^{2\pi} \sqrt{(G_1 - l)^2 + 4\beta} \, da$$

$$+ ½[2\pi \ln(G_1 + l) + \frac{2\sqrt{H}}{G_1 + l} 2\sqrt{K+1} \; E(\pi, q_1) - 2\pi \ln(G_1 - l)$$

$$- \frac{2\sqrt{H}}{G_1 - l} 2\sqrt{K'+1} \; E(\pi, q_1')]$$

$$= ⅛ (G_1 + l) 4\sqrt{H}\sqrt{K+1} E(\pi, q_1) - ⅛ (G_1 - l) \; 4\sqrt{H} \; \sqrt{K'+1} \; E(\pi, q_1')$$

$$+ \pi \ln(G_1 + l) + \frac{2\sqrt{H}}{G_1 + l} \sqrt{K+1} \; E(\pi, q_1) - \frac{2\sqrt{H}}{G_1 - l} \sqrt{K'+1} \; E(\pi, q_1') \quad \ldots(4.10)$$

(using (4.6)(4.7)(4.8)and(4.9))

Now $I_2 = \int_{-l/2}^{l/2} \sqrt{z^2 + G_1 z + B'} \, dz$

$$= \int_{\frac{G_1 - l}{2}}^{\frac{G_1 + l}{2}} \sqrt{u^2 + \beta'} \, du$$

Where $u = z + \frac{G_1}{2}$ and $\beta' = B' - \frac{G_1^2}{4}$

$$= R^2 + H\,Cos\alpha - R^2 Cos\varphi'$$
$$= R^2 \sin^2\varphi' + H\,Cos\alpha$$

$$= \frac{1}{2}\ \{(G_1 + l)\ \sqrt{(G_1 + l)^2 + 4\beta'} - (G_1 - l)\ \sqrt{(G_1 - l)^2 + 4\beta'}\ \} +$$

$$\tfrac{1}{2}\ [\ln (G_1 + l) + \sqrt{(G_1 + l)^2 + 4\beta'} - \ln (G_1 - l) + \sqrt{(G_1 - l)^2 + 4\beta'}\ ]$$

............................(4.11)

Let $D' = R^2 Sin^2\varphi'$

Then $\beta' = D' + HCos\alpha$

$$\sqrt{(G_1 + l)^2 + 4\beta'} = 2\sqrt{H}\ \sqrt{K_1 + Cos\alpha}$$

where, $K_1 = \frac{4D' + (G_1 + l)^2}{4H}$

$$\therefore \int_{-\theta'}^{2\pi-\theta'} I_2\, d\alpha = \tfrac{1}{8}(G_1+l)\int_0^{2\pi}\sqrt{(G_1+l)^2+4\beta'}\, d\alpha - \frac{1}{8}(G_1-l)\int_0^{2\pi}\sqrt{(G_1-l)^2+4\beta'}\, d\alpha$$

$$+\tfrac{1}{2}\left\{\int_0^{2\pi}\ln\left(G_1+l+\sqrt{(G_1+l)^2+4\beta'}\right)d\alpha - \int_0^{2\pi}\ln\left(G_1-l+\sqrt{(G_1-l)^2+4\beta'}\right)d\alpha\right\}$$

$$= \tfrac{1}{8}(G_1+l)\,4\sqrt{H}\,\sqrt{K_1+1}\;E(\pi, q_2) - \tfrac{1}{8}(G_1-l)\,4\sqrt{H}\,\sqrt{K_1'+1}\;E(\pi, q_2^1)$$

$$+\pi\ln(G_1+l) + \frac{2\sqrt{H}}{G_1+l}\sqrt{K_1+1}\;E(\pi, q_2) - \frac{2\sqrt{H}}{G_1-l}\sqrt{K_1'+1}\;E(\pi, q_2^1)$$

.... (4.12)

Where $K_1 = \dfrac{4D' + (G_1+l)^2}{4H}$

$$K_1' = \frac{4D' + (G_1-l)^2}{4H}$$

$$q_2^2 = \frac{2}{K_1+1}$$

$$q_2'^2 = \frac{2}{K_1'+1}$$

$$\sqrt{K_1+1} = \frac{\sqrt{4D' + 4H + (G_1+l)^2}}{2\sqrt{H}}$$

$$\sqrt{K_1'+1} = \frac{\sqrt{4D'+4H+(G_1-l)^2}}{2\sqrt{H}}$$

$$\therefore -\psi = \int_0^{2\pi} I_1\, d\alpha - \int_0^{2\pi} I_2 d\alpha \qquad , (from\, 4.4)$$

$$-\psi = G\rho \left[\begin{array}{l} \frac{1}{2}\left(\sqrt{H}\sqrt{K+1}(G_1+l)E(\pi,q_1) - \sqrt{H}\sqrt{K'+1}(G_1-l)E(\pi,q_1') \right) \\ + 2\pi \ln(G_1+l) + \frac{2\sqrt{H}}{G_1+l}\sqrt{K+1}\; E(\pi,q_1) \\ - 2\pi \ln(G_1-l) - \frac{2\sqrt{H}}{G_1-l}\sqrt{K'+1}\; E(\pi,q_1') \\ - \sqrt{H}\sqrt{K_1'+1}(G_1+l)E(\pi,q_2) \\ + \sqrt{H}\sqrt{K_1+1}(G_1-l)E(\pi,q_2') \\ - 2\pi \ln(G_1+l) - \frac{2\sqrt{H}}{G_1+l}\sqrt{K_1+1}\; E(\pi,q_2) \\ + 2\pi \ln(G_1-l) + \frac{2\sqrt{H}}{G_1-l}\sqrt{K_1'+1}\; E(\pi,q_2') \end{array} \right]$$

using (4.10)and(4.12)

$$or \quad -\psi = G\rho\left[\frac{1}{2}\sqrt{H}\,(G+l)\left\{\sqrt{K+1}\; E(\pi,q_1) - \sqrt{K_1+1}\,E(\pi,q_2)\right\}\right.$$

$$\left. -\frac{1}{2}\sqrt{H}\,(G_1-l)\left\{\sqrt{K'+1}\,E(\pi,q_1^1) - \sqrt{K_1^1+1}\,E(\pi,q_2^1)\right\}\right]$$

$$+\frac{2\sqrt{H}}{G_1+l}\left\{\sqrt{K+1}\,E(\pi,q_1)-\sqrt{K_1+1}\,E(\pi,q_2)\right)$$

$$-\frac{2\sqrt{H}}{G_1-l}\sqrt{K'+1}\,E\left(\pi,q_1'\right)-\sqrt{K_1'+1}\,E\left(\pi,q_2'\right)\Big\}\Big]$$

$$or\quad -\psi=G\rho\left[\left\{\sqrt{K+1}\,E(\pi,q_1)-\sqrt{K_1+1}\,E(\pi,q_2)\right\}\left\{\frac{\sqrt{H}\,(G_1+l)}{2}+\frac{2\sqrt{H}}{G+l}\right\}\right.$$

$$\left.-\left\{\sqrt{K'+1}\;E\left(\pi,q_1'\right)-\sqrt{K_1'+1}\,E\left(\pi,q_2'\right)\right\}\left\{\frac{\sqrt{H}\,(G_1-l)}{2}+\frac{2\sqrt{H}}{G_1-l}\right\}\right]$$

$$=G\rho\sqrt{H}\left[\frac{(G_1+l)^2+4}{2(G_1+l)}\left\{\sqrt{K+1}\,E(\pi,q_1)-\sqrt{K_1+1}\,E(\pi,q_2)\right\}\right.$$

$$\left.-\frac{(G_1-l)^2+4}{2(G_1-l)}\left\{\sqrt{K'+1}\,E\left(\pi,q_1'\right)-\sqrt{K_1'+1}\,E\left(\pi,q_2'\right)\right\}\right]$$

$$or\quad -\psi=G\rho\sqrt{H}\left[\frac{(G_1+l)^2+4}{2(G_1+l)}\left(\frac{\sqrt{4D+4H+(G_1+l)^2}}{2\sqrt{H}}E(\pi,q_1)\right)\right.$$

$$-\frac{\sqrt{4D' + 4H + (G_1 + l)^2}}{2\sqrt{H}} E(\pi, q_2) \Bigg)$$

$$-\frac{(G_1 - l)^2 + 4}{2(G_1 - l)} \left(\frac{\sqrt{4D + 4H + (G_1 - l)^2}}{2\sqrt{H}} E(\pi, q_1') \right.$$

$$\left. -\frac{\sqrt{4D' - 4H + (G_1 - l)^2}}{2\sqrt{H}} E(\pi, q_2') \right)$$

$$\text{or} \quad -\psi = \frac{G\rho}{4} \left\{ \frac{(G_1 + l)^2 + 4}{G_1 + l} \left(\sqrt{4(D + H) + (G_1 + l)^2}\, E(\pi, q_1) \right. \right.$$

$$\left. - \sqrt{4(D' + H) + (G_1 + l)^2}\, E(\pi, q_2) \right) -$$

$$\frac{(G_1 - l)^2 + 4}{G_1 - l} \left(\sqrt{4(D + H) + (G - l)^2}\, E(\pi, q_1') \right) -$$

$$\sqrt{4(D'+H)+(G_1-l)^2}\ E\left(\pi,q_2'\right)\Big)\Big\}\ldots\ldots.\ (4.13)$$

Resolving values of G_1 , D, H *and* D' , *we get*

$$\therefore\ -\psi=\frac{G\rho}{4}\left\{\frac{(-2RCos\varphi'+l)^2+4}{(-2RCos\varphi'+l)}\left(\sqrt{4R_0^2+4R^2+l^2-8RSin\varphi'-4RlCos\varphi'}\ E(\pi,q_1)\right.\right.$$

$$\left.-\sqrt{4R^2+l^2-8RSin\varphi'-4RlCos\varphi'}\ E(\pi,q_2)\right)+$$

$$\frac{(2RCos\varphi'+l)^2+4}{(2RCos\varphi'+l)}\left(\sqrt{4R_0^2+4R^2+l^2-8RSin\varphi'+4RlCos\varphi'}\ E(\pi,q_1')\right.$$

$$\left.\left.-\sqrt{4R^2+l^2-8RSin\varphi'+4RlCos\varphi'}\ E(\pi,q_2')\right)\right\}\ \ldots\ldots\ (4.14)$$

Where

$$E(\pi,q_1)=\pi\left\{1-\frac{1}{2^2}\,q_1^2-\frac{1^2.3}{2^24^2}\,q_1^4-\ldots\frac{(2n-1)!!}{2^n n!}\,\frac{q_1^{2n}}{2n-1}\right\}$$

$$q_1^2=\frac{2}{K+1}=\frac{-16RSin\varphi'}{4R_0^2+4R^2+l^2-4RlCos\varphi'-8RSin\varphi'}$$

$$E(\pi,q_2)=\pi\left\{1-\frac{1}{2^2}\,q_2^2-\frac{1^2.3}{2^24^2}\,q_2^4-\ldots..\frac{(2n-1)!!}{2^n n!}\,\frac{q_2^{2n}}{2n-1}\right\}$$

$$q_2^2 = \frac{2}{K_1+1} = \frac{-16RSin\varphi'}{4R_0^2 + l^2 - 4RlCos\varphi' - 8RSin\varphi'}$$

$$E(\pi, q_1') = \pi\left\{1 - \frac{1}{2^2} q_1'^2 - \frac{1^2.3}{2^2.4^2}\left(q_1'\right)^4 - \dots\dots\dots\dots\right\}$$

$$q_1'^2 = \frac{-16RSin\varphi'}{4R_0^2 + 4R^2 + l^2 + 4RlCos\varphi' - 8RSin\varphi'}$$

$$E(\pi, q_2') = \pi\left\{1 - \frac{1}{2^2} q_2'^2 - \frac{1^2.3}{2^2.4^2}\left(q_2'\right)^4 - \dots\dots\dots\dots\right\}$$

$$q_2'^2 = \frac{-16RSin\varphi'}{4R^2 + l^2 + 4RlCos\varphi' - 8RSin\varphi'}$$

In figure 4.2, ψ vs R is plotted; $\varphi = 0^0$, $30^0, 45^0, 60^0$ and 90^0, Figure 4.3 and 4.4 show the plot of ψ vs φ. These figures (using FORTRAN – LF9556 Compiler) show the variation of the Gravitational Potential with respect to angle φ and distance R.

The programme is given in Appendix-I

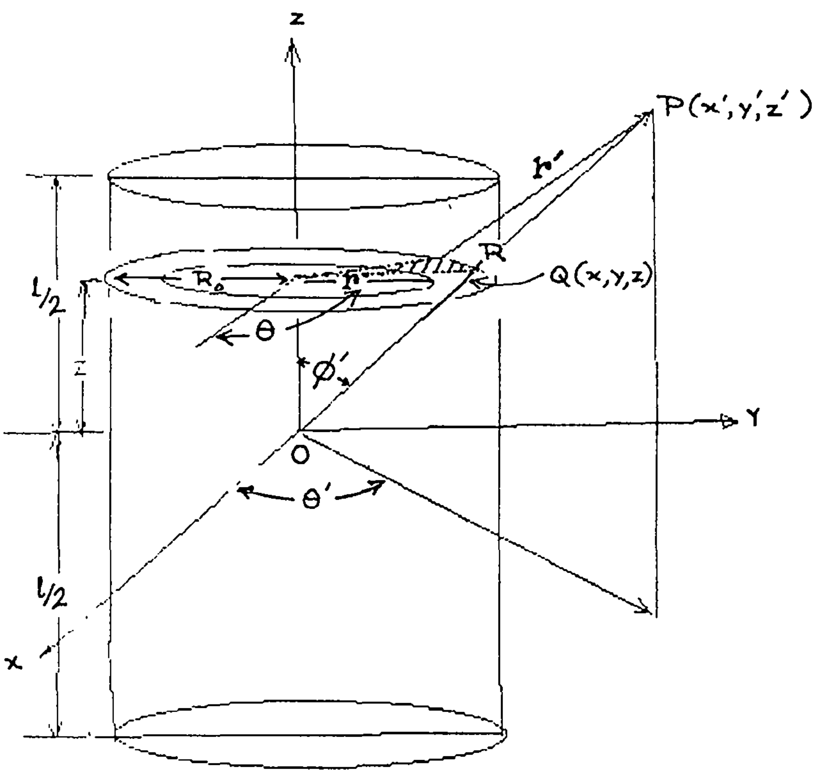

Figure 4.1. Cylindrical mass.

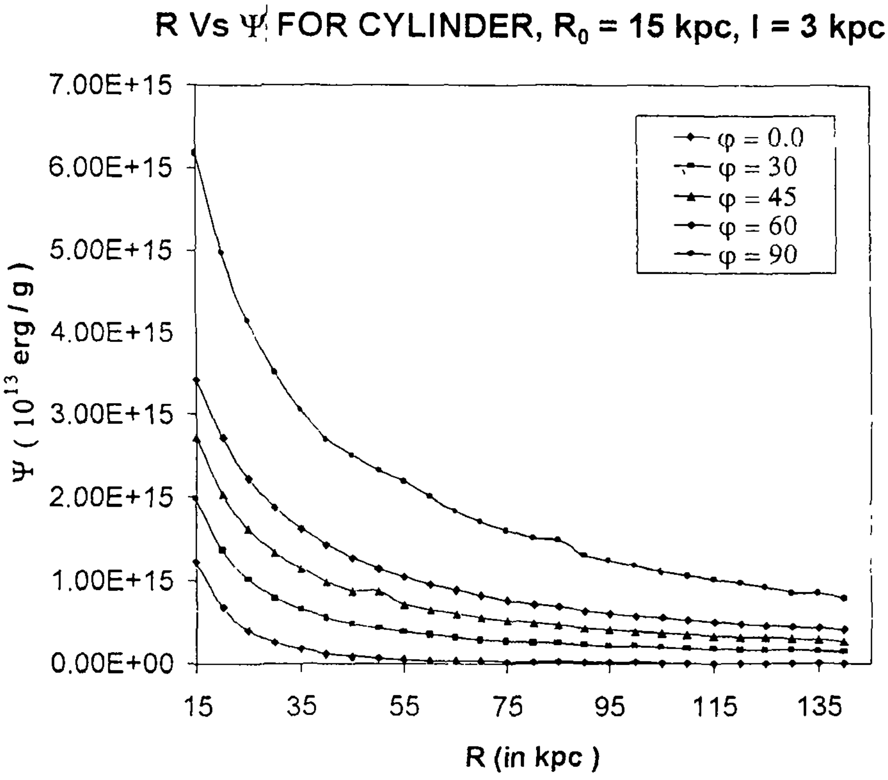

Figure 4.2 Plot of Ψ (10^{13} erg / g) and R (in kpc) for values of Φ = 0^0, 30^0 45^0 60^0 and 90^0.

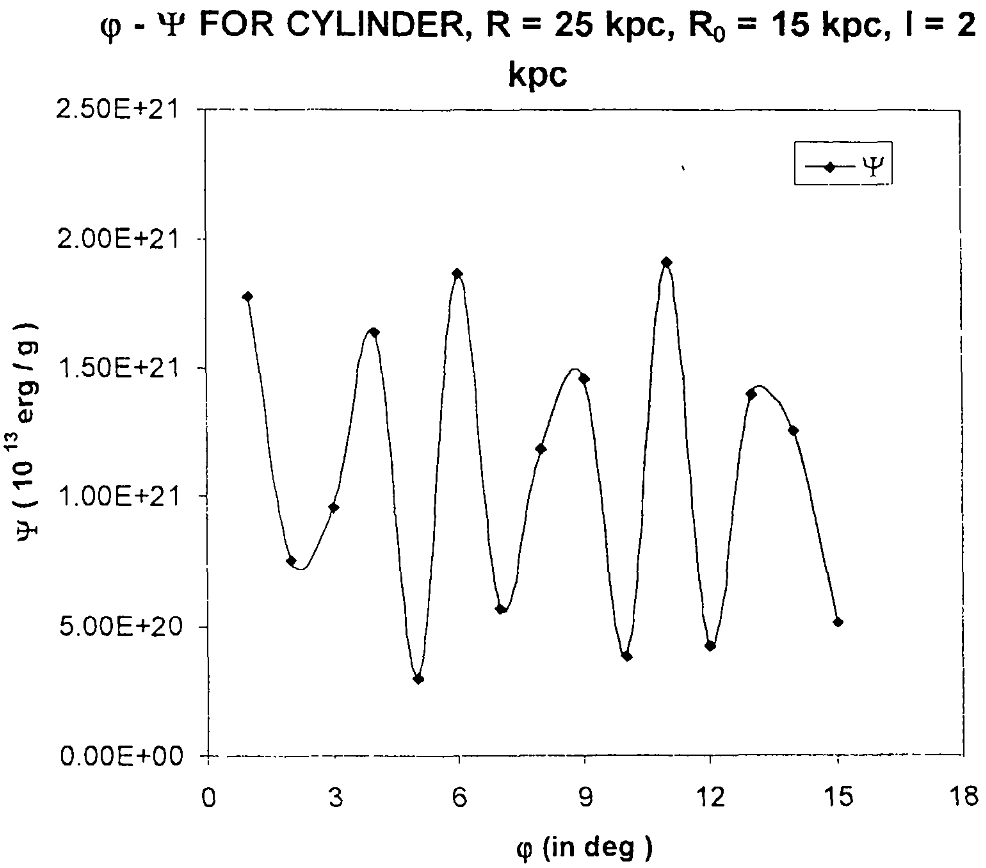

Figure 4.3 Plot of Ψ Vs. φ (in deg.)

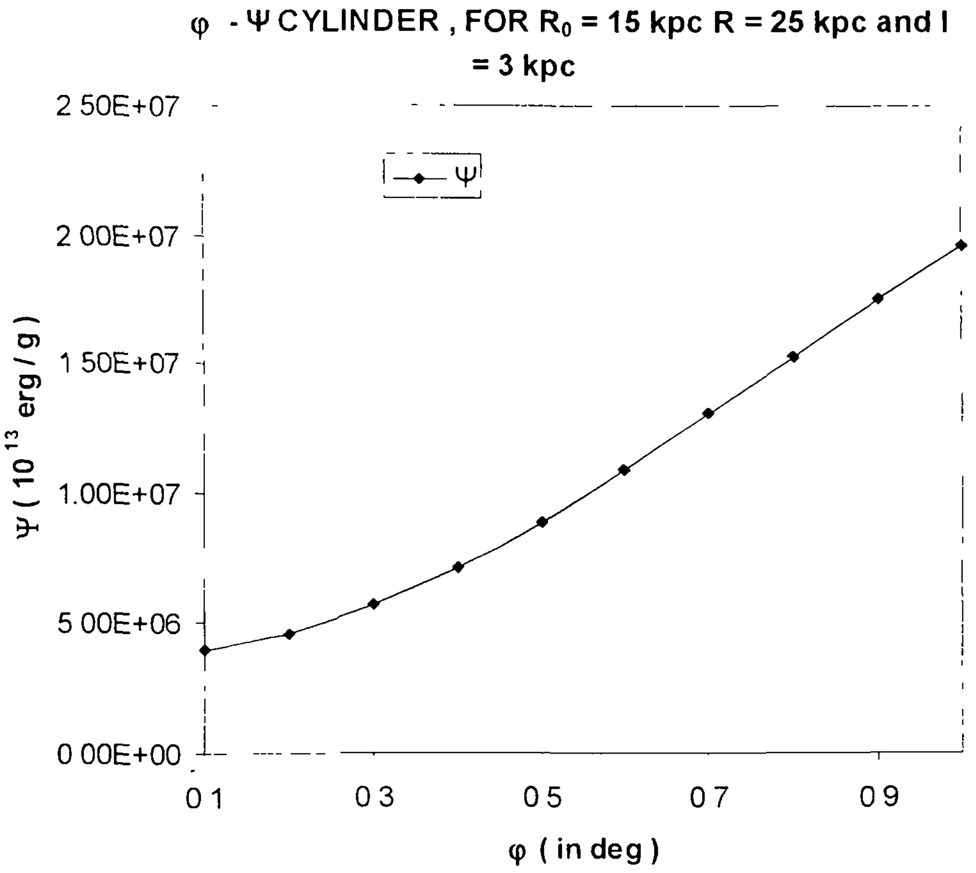

Figure 4.4 Plot of Ψ (10^{13} erg / g) and φ (in deg) for low values of φ.

5

Calculation of Gravitational Potential of an Ellipsoidal Mass of Prolate shape at Any Point outside The Prolate

In this chapter we calculate the gravitational potential of an ellipsoidal mass of prolate shape at any point outside the prolate.

Let us consider a prolate of length l (c.f. figure 5.1). The gravitational potential at a point $P(x', y', z')$ is to be calculated. At first we calculate the gravitational potential due to the elementary disc at a distance z from the centre of the prolater and then it will be integrated for the whole length of the prolate for getting the total potential due to the whole prolate.

We take spherical polar co-ordinate system $P(x',y',z')$ be the point where potential is to be calculated

$$P(x',y',z') = P(R Cos\theta' Sin\varphi', RSin\theta' Sin\varphi', RCos\varphi')$$

$$Q(x',y',z') = Q(rCos\theta, rSin\theta, z) \quad on\ disc$$

$$PQ^2 = (RCos\theta' Sin\varphi' - rCos\theta)^2 + (RSin\theta' Sin\varphi' - rSin\theta)^2 + (RCos\varphi' - z)^2$$

$$r'^2 = R^2 Sin^2\varphi'(Cos^2\theta' + Sin^2\theta') + R^2 Cos^2\varphi' + r^2 + (Cos^2\theta' + Sin^2\theta')$$

$$+ z^2 - 2RSin\varphi'(Cos\theta' Cos\theta - Sin\theta' Sin\theta) - 2Rz\, Cos\varphi'$$

$$= R^2 Sin^2\varphi' + R^2 Cos^2\varphi' + r^2 + z^2 - 2R\, Sin\varphi' Cos(\theta - \theta') - 2Rz\, Cos\varphi'$$

$$= R^2 + r^2 + z^2 - 2rz\, Cos\varphi' - 2RSin\varphi' Cos(\theta - \theta')$$

$$= R^2 + r^2 + z^2 + H\, Cos\alpha + G_1 z$$

Where, $H = -2R\,Sin\varphi'$

$\alpha = \theta - \theta'$

$G_1 = -2R\,Cos\varphi'$

and $d\alpha = d\theta$

Potential at P is

$$d\psi = \frac{-G\rho r dr d\theta dz}{\sqrt{R^2 + r^2 + z^2 + HCos\alpha + G_1 z}} \qquad (\rho \text{ density per unit volume})$$

$$\psi = -G\rho \int_0^b \int_{-\theta'}^{2\pi-\theta'} \int_{\frac{-b^2}{4a}}^{b^2/4a} \frac{r dr d\theta dz}{\sqrt{R^2 + r^2 + z^2 + HCos\alpha + G_1 z}} \quad \text{......................(5.1)}$$

r: $0 \rightarrow b$

$\theta : 0 \rightarrow 2\pi$

α: $-\theta^1 \rightarrow 2\pi - \theta'$

z: $\frac{-b^2}{4a} \rightarrow \frac{b^2}{4a}$

$$\int_0^b \frac{r dr}{\sqrt{A + r^2}} = \sqrt{R^2 + z^2 + HCos\alpha + G_1 z + b^2} - \sqrt{R^2 + z^2 + HCos\alpha + G_1 z}$$

......................(5.2)

$R^2 + b^2 + H\cos\alpha = B$

$R^2 + H\,Cos\alpha = B'$

$$I = \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{z^2 + G_1 z + B}\; dz - \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{z^2 + G_1 z + B'}\, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.3)$$

$$= I' - I''$$

Now $\quad z^2 + G_1 z + B^2 = u^2 + \beta$

$$u = z + \frac{G_1}{2} \quad , \qquad \beta = B - \frac{G_1^{\,2}}{4} \quad , \qquad du = dz$$

as $\quad z \to \dfrac{-b^2}{4a},\ u \to \dfrac{G_1}{2} - \dfrac{b^2}{4a}$

$$z \to \frac{b^2}{4a},\ u \to \frac{G_1}{2} + \frac{b^2}{4a}$$

$$I' = \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{z^2 + G_1 z + B}\, dz$$

$$= \int_{\frac{G_1}{2}-\frac{b^2}{4a}}^{\frac{G_1}{2}+\frac{b^2}{4a}} \sqrt{u^2 + \beta}\; du$$

$$= \frac{1}{2}\left[u\sqrt{u^2 + \beta} + \log\left(u + \sqrt{u^2 + \beta} \right) \right]_{\frac{G_1}{2}-\frac{b^2}{4a}}^{\frac{G_1}{2}+\frac{b^2}{4a}}$$

$$=\frac{1}{2}\left[\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta}+\log\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta}\right\}\right]-$$

$$\frac{1}{2}\left[\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)^2+\beta}+\log\left\{\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)^2+\beta}\right\}\right]$$

$$=I_1+I_2+I_3+I_4$$

Where, $G_1=-2RCos\phi'$,

and $\beta=B-\frac{G^2}{4}=R^2+b^2+HCos\alpha-R^2Cos^2\phi'$

$$=R^2Sin^2\phi'+b^2+HCos\alpha$$

$$=D+HCos\alpha \qquad \left[D=R^2Sin^2\phi'+b^2\right]$$

$$\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta}=\sqrt{\frac{1}{4^2a^2}\left\{\left(2G_1a+b^2\right)^2+4^2a^2\beta\right\}}$$

$$=\frac{1}{4a}\sqrt{\left\{\left(2G_1a+b^2\right)^2+4^2a^2\left(D+HCos\alpha\right)\right\}}$$

$$=\frac{1}{4a}\sqrt{\left(2G_1a+b^2\right)^2+4^2a^2D+4^2a^2HCos\alpha}$$

$$=\frac{4a\sqrt{H}}{4a}\sqrt{\frac{\left(2G_1a+b^2\right)+4^2a^2D}{4^2a^2H}+Cos\alpha}$$

$$=\sqrt{H}\sqrt{K+Cos\alpha},$$

Where, $K = \frac{\left(2G_1 a + b^2\right)^2 + 4^2 a^2 D}{4^2 a^2 H} = \frac{1}{H}\left[\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + D\right]$

$$I_1 = \int_{-\theta'}^{2\pi-\theta'} \left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + \beta}\ d\alpha$$

$$= \left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\int_{-\theta'}^{2\pi-\theta'} \sqrt{H}\sqrt{K + Cos\alpha}\ d\alpha$$

$$= \left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\sqrt{H}\int_{-\theta'}^{2\pi-\theta'} \sqrt{K + Cos\alpha}\ d\alpha$$

$$= \left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\sqrt{H}\sqrt{K+1}\ \ 2E\left(\pi, q_1\right), \text{ where, } q_1^2 = \frac{2}{K+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(5.4)}$$

Calculation of $\int_{-\theta'}^{2\pi-\theta'} \sqrt{K + Cos\alpha}\ d\alpha$,

using $\int\sqrt{a + bCosx} = \sqrt{a+b}\int\sqrt{1 - \frac{2b}{a+b}.Sin^2\frac{x}{2}}\ dx$

we get, $\int_{-\theta'}^{2\pi-\theta'} \sqrt{K + Cos\alpha}\ d\alpha$

$$= \sqrt{K+1}\int_{-\theta'}^{2\pi-\theta'} \sqrt{1 - q_1^2 Sin^2\frac{\alpha}{2}}\ d\alpha$$

where, $K = a$, $b = 1$, $q_1^2 = \frac{2}{K+1}$ and $\frac{\alpha}{2} = \gamma$, $d\alpha = d\gamma$, considering x in the direction of P, $\theta' = 0$, $\alpha = \theta'$ *and as* $\alpha \to 0$ *to* 2π, $\gamma \to 0$ *to* π, *then*

$$\int_{-\theta}^{2\pi-\theta''} \sqrt{1-q_1^2 Sin^2 \frac{\alpha}{2}}\, d\alpha = 2\int_0^{\pi} \sqrt{1-q_1^2 Sin\gamma}\;\; d\gamma$$

$$=2E(\pi, q_1).$$

$$I_2 = \int_{-\theta'}^{2\pi-\theta'} \log\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta}\right\}\, d\alpha$$

$$= \int_0^{2\pi} \ln\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\sqrt{H}\sqrt{K+Cos\alpha}\right\}\, d\alpha$$

$$= \int_0^{2\pi} \ln\sqrt{H}\left(\frac{2G_1 a+b^2}{4a\sqrt{H}}+\sqrt{K+Cos\alpha}\right)\, d\alpha$$

$$= \int_0^{2\pi} \ln\sqrt{H}\, d\alpha + \int_0^{2\pi} \ln\frac{2G_1 a+b^2}{4a\sqrt{H}} d\alpha + \int_0^{2\pi} \ln\left(1+\frac{4a\sqrt{H}}{2G_1 a+b^2}\sqrt{K+Cos\alpha}\right) d\alpha$$

$$\left(as \;\; \log(1+x) = x - \frac{x^2}{2}+\frac{x^3}{3} - \;\; \ldots \quad when\; |x| \le 1\right)$$

$$= 2\pi \ln\sqrt{H} + \int_0^{2\pi} \ln\frac{1}{\sqrt{H}} d\alpha \; + \; 2\pi \ln\left(\frac{G_1}{2}+\frac{b^2}{4a}\right) + \int_0^{2\pi} \frac{4a\sqrt{H}}{2G_1 a+b^2}\sqrt{K+Cos\alpha}\, d\alpha$$

$$=2\pi \ln\left(\frac{G_1}{2}+\frac{b^{2}}{4a}\right)+\frac{4a\sqrt{H}}{2G_1 a+b^2}\int_0^{2\pi} \sqrt{K+Cos\alpha}\; d\alpha$$

$$= 2\pi \ln\left(\frac{G_1}{2} + \frac{b^2}{4a}\right) + \frac{4a\sqrt{H}}{2G_1 a + b^2}\sqrt{K+1}\; 2E(\pi, q_1) \quad \ldots\ldots\ \ldots\ldots\ldots\ldots\ldots (5.5)$$

$$I_3 = \int_0^{2\pi}\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta}\; d\alpha$$

Now, $\sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta} = \sqrt{H}\sqrt{K_1 + Cos\alpha}\,, \quad K_1 = \frac{\left(2G_1 a - b^2\right)^2 + 4^2 a^2 D}{4^2 a^2 H}$

$$\therefore \quad I_3 = \left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{H}\int_0^{2\pi}\sqrt{K_1 + Cos\alpha}\; d\alpha$$

$$= \left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{H}\sqrt{K_1 + 1}\; 2E(\pi, q_2)\,; \qquad q_2^2 = \frac{2}{K_1 + 1} \quad \ldots\ldots\ldots\ldots (5.6)$$

$$I_4 = \int_{-\theta'}^{2\pi - \theta'} \ln\left\{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right) + \sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta}\right\} d\alpha$$

$$= 2\pi \ln\left(\frac{G_1}{2} - \frac{b^2}{4a}\right) + \frac{4a\sqrt{H}}{2G_1 a - b^2}\sqrt{K_1 + 1}\; 2E(\pi, q_2') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.7)$$

Now, $I'' = \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{z'^2 + Gz + B'}\; dz$

$$= \int_{\frac{G_1}{2}-\frac{b^2}{4a}}^{\frac{G_1}{2}+\frac{b^2}{4a}} \sqrt{u^2 + \beta'} \; du$$

Where, $u = z + \frac{G_1}{2}$

and

$$\beta' = B' - \frac{G_1^2}{4} = R^2 + HCos\alpha - R^2Cos^2\phi = R^2Sin^2\phi' + HCos\alpha = D' + HCos\alpha$$

where, $D' = R^2Sin^2\phi'$

when, $z \to -\frac{b^2}{4a}$, $u \to \frac{G_1}{2} - \frac{b^2}{4a}$

and $z \to \frac{b^2}{4a}$, $u \to \frac{G_1}{2} + \frac{b^2}{4a}$

Now, $$I'' = \frac{1}{2}\left[\int_{-\theta'}^{2\pi-\theta'} \left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + \beta'} \; d\alpha \right.$$

$$\left. - \int_{-\theta'}^{2\pi-\theta'} \left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta'} \; d\alpha \right.$$

$$+\int_{-\theta'}^{2\pi-\theta'} \ln\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta'}\right\}d\alpha$$

$$-\int_{-\theta'}^{2\pi-\theta'} \log\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)^2+\beta'}\ d\alpha\Bigg] \quad\ldots\ldots\ldots\ldots\ldots\ldots (5.8)$$

Or, $$I''=I_1''-I_2''+I_3''-I_4'' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.9)$$

where,

$$I_1'' = \int_{-\theta'}^{2\pi-\theta'}\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta'}\ d\alpha$$

$$=\int_{-\theta'}^{2\pi-\theta'}\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+D'+HCos\alpha}\ d\alpha$$

$$=\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{H}\int_{-\theta'}^{2\pi-\theta'}\sqrt{\frac{1}{H}\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+D'+Cos\alpha\right\}}\ d\alpha$$

$$= \left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{H}\int_{r_1}^{2\pi}\sqrt{K'+Cos\alpha}\;d\alpha \qquad where,\; K' = \frac{1}{H}\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2 + D'\right\}$$

$$= \left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{H}\,\sqrt{K'+1}\;\; 2E\left(\pi, q_1'\right),\; where, \quad q_1' = \frac{2}{K'+1} \qquad \ldots\ldots \qquad (5\ 10)$$

$$I_2'' = \int_{-\theta'}^{2\pi-\theta'}\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)^2 + \beta'}\; d\alpha$$

$$= \left(\frac{G_1}{2}-\frac{b^2}{4a}\right)\sqrt{H}\,\sqrt{K_1'+1}\;\; 2E\left(\pi, q_2'\right),\; where,\; K_1' = \frac{\left(2aG_1-b^2\right)^2 + 4a^2D'}{4^2a^2H}$$

$$\text{and} \quad q_2' = \frac{2}{K_1'+1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.11)$$

$$I_3'' = \int_{-\theta'}^{2\pi-\theta'} \ln\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right) + \sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2 + \beta'}\right\} d\alpha$$

$$= 2\pi\ln\left(\frac{G_1}{2}+\frac{b^2}{4a}\right) + \frac{4a\sqrt{H}}{2G\,a+b^2}\sqrt{K'+1}\;\; 2E\left(\pi, q_1'\right)\ldots \qquad \ldots\ldots\ldots\ldots \qquad (5.12)$$

$$I_4'' = 2\pi \ln\left(\frac{G_1}{2} - \frac{b^2}{4a}\right) + \frac{4a\sqrt{H}}{2G_1 a - b^2}\sqrt{K_1' + 1}\ \ 2E\left(\pi, q_2'\right) \quad \ldots \quad . \quad \ldots .\ (5\ 13)$$

$$\therefore \quad I' = \pi \ln \frac{2G_1 a + b^2}{2G_1 a - b^2} +$$

$$\sqrt{H}\left\{\left(\frac{2G_1 a + b^2}{4a} + \frac{4a}{2G_1 a + b^2}\right)\sqrt{K+1}\ E(\pi, q_1) + \left(\frac{2G_1 a - b^2}{4a} + \frac{4a}{2G_1 a - b^2}\right)\sqrt{K_1 + 1}\ E(\pi, q_2)\right\}$$

and

$$I'' = \pi \ln \frac{2G_1 a + b^2}{2G_1 a - b^2} +$$

$$\sqrt{H}\left\{\left(\frac{2G_1 a + b^2}{4a} + \frac{4a}{2G_1 a + b^2}\right)\sqrt{K'+1}\,E\left(\left(\pi, q_1'\right) - \left(\frac{2G_1 a - b^2}{4a} + \frac{4a}{2G_1 a - b^2}\right)\sqrt{K_1' + 1}\,E\left(\pi, q_2'\right)\right)\right\}$$

$$I = I' - I''$$

$$= G\rho \left[\begin{array}{l} \left(\frac{2G_1 a + b^2}{4a} + \frac{4a}{2G_1 a + b^2}\right)\left\{\sqrt{H}\sqrt{K+1}\ E(\pi, q_1) - \sqrt{H}\ \sqrt{K'+1}\,E\left(\pi, q_1\right)\right\} \\ \\ -\left(\frac{2G_1 a - b^2}{4a} + \frac{4a}{2G_1 a - b^2}\right)\left\{\sqrt{H}\sqrt{K_1 + 1}\,E(\pi, q_2) - \sqrt{H}\sqrt{K_1' + 1}\ E\left(\pi, q_2'\right)\right\} \end{array}\right] \quad \ldots (5\ 14)$$

When, $E(\pi, q) = \pi\left\{1 - \frac{1^2}{2^2}q^2 - \frac{1^2.3}{2^2.4}q^4 - \quad \ldots \quad \ldots\ldots\right\}$

Now,

$$\sqrt{H}\sqrt{K+1} = \sqrt{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + D + H}$$

$$= \sqrt{\left(-RCos\phi' + \frac{b^2}{4a}\right)^2 + R^2 Sin^2\phi' + b^2 - 2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2 + 16a^2b^2 + b^4 - 8ab^2RCos\varphi' - 32a^2RSin\varphi'}$$

Now,

$$\sqrt{H}\sqrt{K'+1} = \sqrt{\left(-RCos\phi' + \frac{b^2}{4a}\right)^2 + R^2 Sin^2\phi' - 2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2 + b^4 - 8ab^2RCos\varphi' - 32a^2RSin\varphi'} \,,$$

$$\sqrt{H}\sqrt{K_1+1} = \sqrt{\left(-RCos\phi' - \frac{b^2}{4a}\right)^2 + R^2 Sin^2\phi + b^2 - 2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2 + 16a^2b^2 + b^4 + 8ab^2RCos\varphi' - 32a^2RSin\varphi'}$$

and

$$\sqrt{H}\sqrt{K_1' + 1} = \sqrt{\left(-RCos\phi' - \frac{b^2}{4a}\right)^2 + R^2Sin^2\phi' - 2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2 + b^4 + 8ab^2RCos\varphi' - 32a^2RSin\varphi'}$$

Hence, finally we get

$$\psi = -\frac{G\rho}{4a}\left[\left(\frac{-4aRCos\varphi' + b^2}{4a} + \frac{4a}{-4aRCos\varphi' + b^2}\right)\times\right.$$

$$\left\{\sqrt{16a^2R^2 + 16a^2b^2 + b^4 - 8ab^2RCos\varphi' - 32a^2RSin\varphi'}\; E(\pi, q_1)\right.$$

$$\left.-\sqrt{16a^2R^2 + b^4 - 8ab^2RCos\varphi' - 32a^2RSin\varphi'}\; E(\pi, q_1')\right\}$$

$$+\left(\frac{4aRCos\varphi' + b^2}{4a} + \frac{4a}{4aRCos\varphi' + b^2}\right)\times$$

$$\left\{\sqrt{16a^2R^2 + 16a^2b^2 + b^4 + 8ab^2RCos\varphi' - 32a^2RSin\varphi'}\; E(\pi, q_2)\right)$$

$$-\sqrt{16a^2R^2+b^4+8ab^2RCos\varphi'-32a^2RSin\varphi'}\;E(\pi,q_2')\}\Big]$$

$$\qquad\ldots\ldots\ldots\ldots(5.15)$$

Where,

$$q_1^2=\frac{-64a^2RSin\varphi'}{16a^2R^2+16a^2b^2+b^4-8ab^2RCos\varphi'-32a^2RSin\varphi'}$$

$$q_1'^2=\frac{-64a^2RSin\varphi'}{16a^2R^2+b^4-8ab^2RCos\varphi'-32a^2RSin\varphi'}$$

$$q_2^2=\frac{-16a^2RSin\varphi'}{16a^2R^2+16a^2b^2+b^4-8ab^2RCos\varphi'-32a^2RSin\varphi'}$$

$$q_2'^2=\frac{-64a^2RSin\varphi'}{16a^2R^2+b^4+8ab^2RCos\varphi'-32a^2RSin\varphi'}$$

Thus, equation (5.15) is the expression for Gravitational Potential of an elliptical mass of Prolate shape at any point outside the Prolate. In Chapter-3 (Sec. 3.3), in case of the calculation of Gravitational Potential of thin uniform straight rod at any point, we have seen that if we draw an ellipse with A and B as foci then for all points on the ellipse, the sum of the focal distances will be the same (where for the point P on the surface of the ellipse the sum of the focal distances is (r_1+r_2)) and for all points over the surface of the ellipse the potential is constant. The potential is also constant over the ellipsoid obtained by revolving this ellipse about the line joining the foci, i.e., about AB.

In figure 5.2, ψ vs R is plotted; $\varphi =$ $30^0, 45^0$ and 90^0, Figure 5.3 shows the plot of ψ vs φ.These figures (using FORTRAN – LF9556 Compiler) show the variation of the Gravitational Potential with respect to angle φ and distance R. In Figure 5.3, we see that the value of ψ drops to a minimum and then rises again. This may be because of the fact that the potential remains constant only in some ellipsoidal surface like in case of an uniform rod as discussed above, whereas in this case the distance R is kept constant and it will generate a spherical surface.

Thus we find that the expression for Gravitational Potential of an elliptical mass of Prolate shape at any point outside the Prolate (equation (5.15)) is correct.

The programme is given in Appendix-II & III.

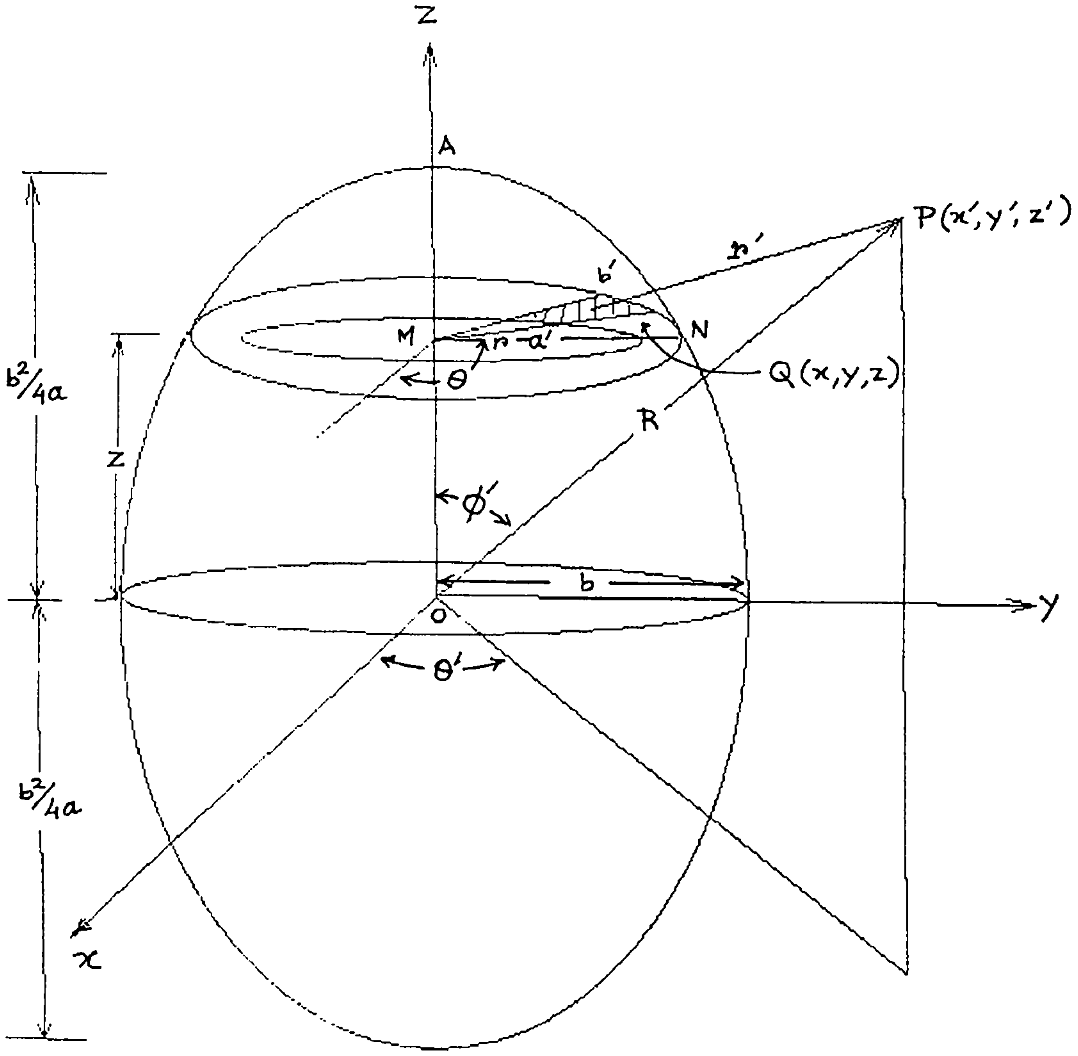

Figure 5.1 Prolate Shaped Mass

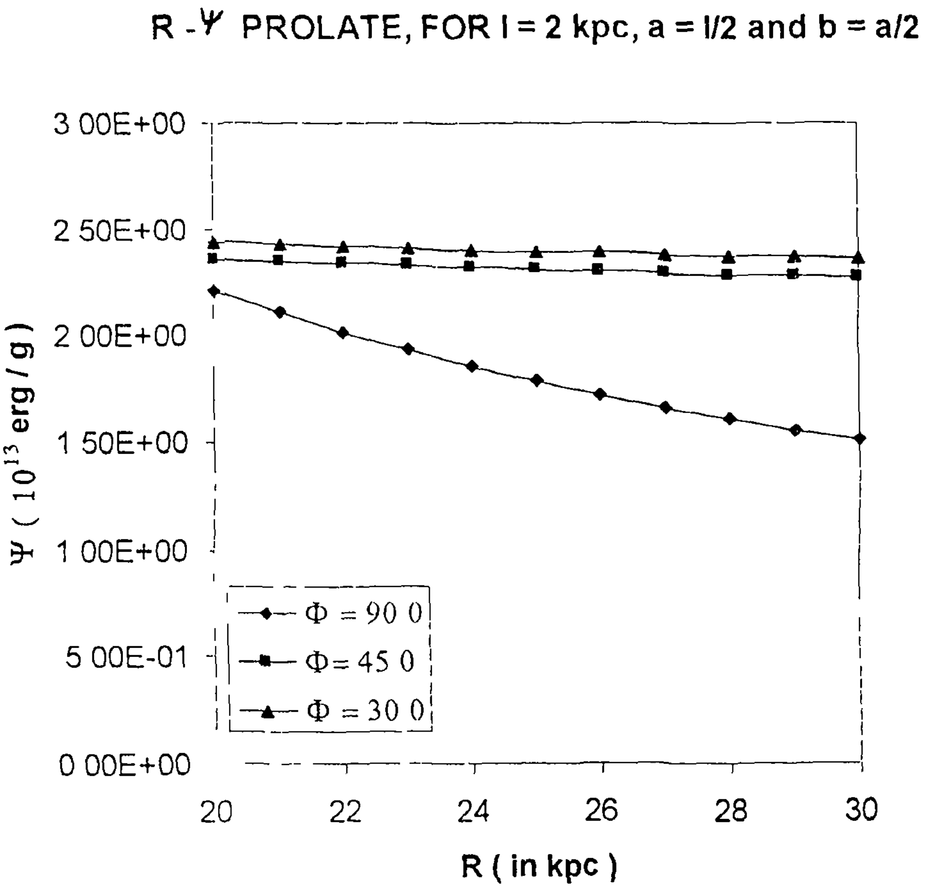

Figure 5.2 Plot of Ψ (10^{13} erg / g) and R (in kpc) for Prolate

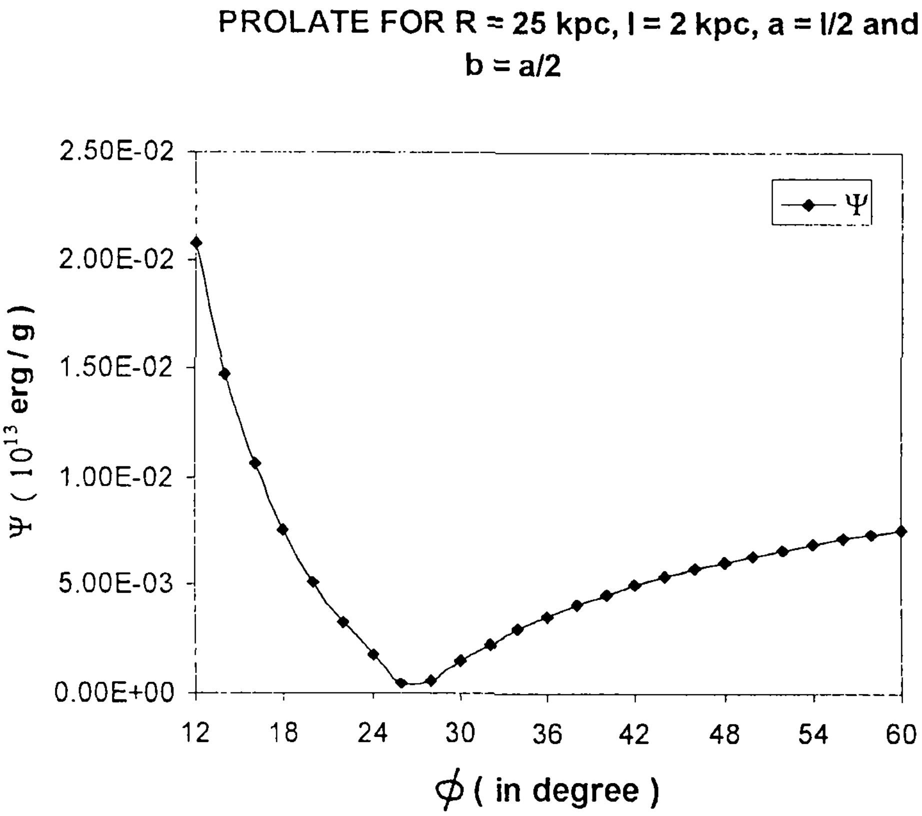

Figure 5.3 Plot of Ψ (10^{13} erg / g) and Φ (in degree) for Prolate

6

Calculation of Gravitational Potential of an Ellipsoidal Mass of Oblate shape at AnyPoint outside The Oblate

In this chapter we calculate the gravitational potential of an ellipsoidal mass of oblate shape at any point outside the oblate.

Let us consider a oblate of length l (c.f. figure 6.1). The gravitational potential at a point $P(x',y',z')$ is to be calculated. At first we calculate the gravitational potential due to the elementary disc at a distance z from the centre of the oblate and then it will be integrated for the whole length of the oblate for getting the total potential due to the whole oblate. As the calculation of Potential for the case of oblate is similar to that of prolate (detailed calculation is done in Chapter-5), in the following we write only the final results:

We take spherical polar co-ordinate system. $P(x',y',z')$ be the point where potential is to be calculated

$$P(x',y',z') = P(RCos\theta' Sin\varphi', RSin\theta' Sin\varphi', RCos\varphi')$$

$$Q(x',y',z') = Q(rCos\theta, rSin\theta, z) \quad on\ disc$$

$$PQ^2 = (RCos\theta' Sin\varphi' - rCos\theta)^2 + (RSin\theta' Sin\varphi' - rSin\theta)^2 + (RCos\varphi' - z)^2$$

$$r'^2 = R^2 Sin^2\varphi'(Cos^2\theta' + Sin^2\theta') + R^2 Cos^2\varphi' + r^2 + (Cos^2\theta' + Sin^2\theta')$$

$$+ z^2 - 2RSin\varphi'(Cos\theta' Cos\theta - Sin\theta' Sin\theta) - 2Rz\,Cos\varphi'$$

$$= R^2 + r^2 + z^2 + H\,Cos\alpha + G_1 z$$

Where, $H = -2R\,Sin\varphi'$

$\alpha = \theta - \theta'$

$G_1 = -2R\,Cos\varphi'$

and $d\alpha = d\theta$

Potential at P is

$$d\psi = \frac{-G\rho r dr d\theta dz}{\sqrt{R^2 + r^2 + z^2 + HCos\alpha + G_1 z}}$$ (ρ density per unit volume)

$$\psi = -G\rho \int_0^b \int_{-\theta'}^{2\pi-\theta'} \int_{\frac{-b^2}{4a}}^{\frac{b^2}{4a}} \frac{r dr d\theta dz}{\sqrt{R^2 + r^2 + z^2 + HCos\alpha + G_1 z}} \qquad \text{.........................(6.1)}$$

(here , $a \succ b$)

r:0 →b

θ :0→2 π

α: - θ' →2 π - θ'

z: $\frac{-b^2}{4a} \to \frac{b^2}{4a}$

$$\int_o^b \frac{r dr}{\sqrt{A + r^2}} = \sqrt{R^2 + z^2 + HCos\alpha + G_1 z + b^2} - \sqrt{R^2 + z^2 + HCos\alpha + G_1 z}$$

..................................(6.2)

$R^2 + b^2 + H\cos\alpha = B$

$R^2 + H\,Cos\alpha = B'$

$$I = \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{z^2 + G_1 z + B}\, dz - \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{z^2 + G_1 z + B'}\, dz$$

$$= I' - I'' \quad \ldots (6\,3)$$

Now $z^2 + G_1 z + B^2 = u^2 + \beta$

$$u = z + \frac{G_1}{2} \quad , \quad \beta = B - \frac{G_1^2}{4} \quad , \quad du = dz$$

as $\quad z \to \frac{-b^2}{4a}, \; u \to \frac{G_1}{2} - \frac{b^2}{4a}$

$\quad z \to \frac{b^2}{4a}, \; u \to \frac{G_1}{2} + \frac{b^2}{4a}$

$$I' = \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{z^2 + G_1 z + B}\, dz$$

$$= \frac{1}{2}\left[\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + \beta} + \log\left\{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right) + \sqrt{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + \beta}\right\}\right] -$$

$$\frac{1}{2}\left[\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta} + \log\left\{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right) + \sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta}\right\}\right]$$

$$= I_1 + I_2 + I_3 + I_4$$

Where, $G_1 = -2RCos\phi'$,

and $\beta = B - \frac{G_1^2}{4} = R^2 + b^2 + HCos\alpha - R^2Cos^2\phi'$

$= R^2Sin^2\phi' + b^2 + HCos\alpha$

$= D + HCos\alpha|$; $\quad [D = R^2Sin^2\phi' + b^2]$

$$\sqrt{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + \beta} = \sqrt{\frac{1}{4^2a^2}\left\{\left(2G_1a + b^2\right)^2 + 4^2a^2\beta\right\}}$$

$$= \frac{1}{4a}\sqrt{\left\{\left(2G_1a + b^2\right)^2 + 4^2a^2\left(D + HCos\alpha\right)\right\}}$$

$$= \sqrt{H}\sqrt{K + Cos\alpha},$$

Where, $K = \frac{\left(2G_1a + b^2\right)^2 + 4^2a^2D}{4^2a^2H} = \frac{1}{H}\left[\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + D\right]$

$$I_1 = \int_{-\theta'}^{2\pi-\theta'}\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} + \frac{b^2}{4a}\right)^2 + \beta}\; d\alpha$$

$$= \left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\int_{-\theta'}^{2\pi-\theta'}\sqrt{H}\sqrt{K + Cos\alpha}\; d\alpha$$

$$= \left(\frac{G_1}{2} + \frac{b^2}{4a}\right)\sqrt{H}\sqrt{K+1}\; 2E(\pi, q_1), \quad \text{where, } q_1^2 = \frac{2}{K+1} \quad \ldots\ldots \quad (6.4)$$

Calculation of $\int_{-\theta'}^{2\pi-\theta'} \sqrt{K+Cos\alpha}\ d\alpha$,

using $\int \sqrt{a+bCosx} = \sqrt{a+b}\int \sqrt{1-\frac{2b}{a+b}.Sin^2\frac{x}{2}}\ dx$

we get, $\int_{\theta'}^{2\pi-\theta'} \sqrt{K+Cos\alpha}\ d\alpha|$

$$= \sqrt{K+1}\int_{-\theta'}^{2\pi-\theta'} \sqrt{1-q_1^2 Sin^2\frac{\alpha}{2}}\ d\alpha$$

where, $K = a,\ b = 1,\ q_1^2 = \frac{2}{K+1}$ and $\frac{\alpha}{2} = \gamma,\quad d\alpha = d\gamma$, considering x in the direction of P, $\theta' = 0,\quad \alpha = \theta\ and\ as\ \alpha \to 0\ to\ 2\pi,\ \gamma \to 0\ to\ \pi,\ then$

$$\int_{-\theta}^{2\pi-\theta'} \sqrt{1-q_1^2 Sin^2\frac{\alpha}{2}}\ d\alpha = 2\int_0^{\pi} \sqrt{1-q_1^2 Sin\gamma}\ \ d\gamma$$

$$= 2E(\pi, q_1).$$

$$I_2 = \int_{-\theta'}^{2\pi-\theta'} \log\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta}\right\}\ d\alpha$$

$$= \int_0^{2\pi} \ln\sqrt{H}\left(\frac{2G_{1a}+b^2}{4a\sqrt{H}}+\sqrt{K+Cos\alpha}\right)\ d\alpha$$

$$= \int_0^{2\pi} \ln\sqrt{H}\, d\alpha + \int_0^{2\pi} \ln\frac{2G_1 a + b^2}{4a\sqrt{H}} d\alpha + \int_0^{2\pi} \ln\left(1 + \frac{4a\sqrt{H}}{2G_1 a + b^2}\sqrt{K + Cos\alpha}\right) d\alpha$$

$$\left(as \ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ \cdots \quad when\ |x| \le 1 \right),$$

$$= 2\pi \ln\sqrt{H} + \int_0^{2\pi} \ln\frac{1}{\sqrt{H}} d\alpha \ + \ 2\pi \ln\left(\frac{G_1}{2} + \frac{b^2}{4a}\right) + \int_0^{2\pi} \frac{4a\sqrt{H}}{2G_0 a + b^2}\sqrt{K + Cos\alpha}\, d\alpha$$

$$= 2\pi \ln\left(\frac{G_1}{2} + \frac{b^2}{4a}\right) + \frac{4a\sqrt{H}}{2G_1 a + b^2}\int_0^{2\pi} \sqrt{K + Cos\alpha}\ d\alpha$$

$$= 2\pi \ln\left(\frac{G_1}{2} + \frac{b^2}{4a}\right) + \frac{4a\sqrt{H}}{2G_1 a + b^2}\sqrt{K+1}\ 2E(\pi, q_1) \quad \ldots\ldots \quad (6.5)$$

$$I_3 = \int_0^{2\pi}\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta}\ d\alpha$$

Now, $\sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta} = \sqrt{H}\sqrt{K_1 + Cos\alpha}, \quad K_1 = \frac{\left(2G_1 a - b^2\right)^2 + 4^2 a^2 D}{4^2 a^2 H}$

$$\therefore \quad I_3 = \left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{H}\int_0^{2\pi}\sqrt{K_1 + Cos\alpha}\ d\alpha$$

$$= \left(\frac{G_1}{2} - \frac{b^2}{4a}\right)\sqrt{H}\sqrt{K_1 + 1}\ \ 2E(\pi, q_2), \quad q_2^2 = \frac{2}{K_1 + 1} \quad \ldots\ldots\ldots\ldots (6.6)$$

$$I_4 = \int_{-\theta'}^{2\pi-\theta'} \ln\left\{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right) + \sqrt{\left(\frac{G_1}{2} - \frac{b^2}{4a}\right)^2 + \beta}\right\} d\alpha$$

$$= 2\pi \ln\left(\frac{G_1}{2} - \frac{b^2}{4a}\right) + \frac{4a\sqrt{H}}{2G_1 a - b^2}\sqrt{K_1 + 1}\; 2E(\pi, q_2) \qquad \text{.........................(6.7)}$$

Now, $I'' = \int_{-\frac{b^2}{4a}}^{\frac{b^2}{4a}} \sqrt{Z^2 + GZ + B'}\; dz$

$$= \int_{\frac{G_1}{2}-\frac{b^2}{4a}}^{\frac{G_1}{2}+\frac{b^2}{4a}} \sqrt{u^2 + \beta'}\; du$$

Where, $u = z + \frac{G_1}{2}$

and

$$\beta' = B' - \frac{G_1^2}{4} = R^2 + HCos\alpha - R^2Cos^2\phi = R^2Sin^2\phi' + HCos\alpha = D' + HCos\alpha$$

where, $D' = R^2 Sin^2\phi'$

when, $z \to -\frac{b^2}{4a}$, $u \to \frac{G_1}{2} - \frac{b^2}{4a}$

and $z \to \frac{b^2}{4a}$, $u \to \frac{G_1}{2} + \frac{b^2}{4a}$

Now, $$I'' = \frac{1}{2}\left[\int_{-\theta'}^{2\pi-\theta'}\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta'}\,d\alpha\right.$$

$$-\int_{-\theta'}^{2\pi-\theta'}\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta'}\,d\alpha$$

$$+\int_{-\theta'}^{2\pi-\theta'}\ln\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta'}\right\}d\alpha$$

$$\left.-\int_{-\theta'}^{2\pi-\theta'}\log\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)^2+\beta'}\;d\alpha\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.8)$$

Or, $$I'' = I_1'' - I_2'' + I_3'' - I_4'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.9)$$

where,

$$I_1'' = \int_{-\theta'}^{2\pi-\theta'}\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta'}\,d\alpha$$

$$= \int_{-\theta'}^{2\pi-\theta'} \left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2 + D' + HCos\alpha}\; d\alpha$$

$$= \left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{H}\int_{-\theta'}^{2\pi-\theta'} \sqrt{\frac{1}{H}\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2 + D' + Cos\alpha\right\}}\; d\alpha$$

$$= \left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{H}\int_{0}^{2\pi} \sqrt{K' + Cos\alpha}\; d\alpha \qquad where,\ K' = \frac{1}{H}\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2 + D'\right\}$$

$$= \left(\frac{G_1}{2}+\frac{b^2}{4a}\right)\sqrt{H}\sqrt{K'+1}\; 2E\left(\pi, q_1'\right), \quad where,\ q_1' = \frac{2}{K'+1} \quad . \qquad (6\ 10)$$

$$I_2'' = \int_{-\theta'}^{2\pi-\theta'} \left(\frac{G_1}{2}-\frac{b^2}{4a}\right)\sqrt{\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)^2 + \beta'}\; d\alpha$$

$$= \left(\frac{G_1}{2}-\frac{b^2}{4a}\right)\sqrt{H}\sqrt{K_1'+1}\; 2E\left(\pi, q_2\right), \qquad where,\ k_1' = \frac{\left(2aG_1 - b^2\right)^2 + 4a^2D'}{4^2a^2H}$$

$$\text{and} \quad q_2' = \frac{2}{K_1'+1}$$

.... (6.11)

$$I_3'' = \int_{-\theta'}^{2\pi-\theta'} \ln\left\{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\sqrt{\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)^2+\beta'}\right\}d\alpha$$

$$= 2\pi\ln\left(\frac{G_1}{2}+\frac{b^2}{4a}\right)+\frac{4a\sqrt{H}}{2G_1a+b^2}\sqrt{K'+1}\ 2E\left(\pi,q_1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.12)$$

$$I_4'' = 2\pi\ln\left(\frac{G_1}{2}-\frac{b^2}{4a}\right)+\frac{4a\sqrt{H}}{2G_1a-b^2}\sqrt{K_1'+1}\ \ 2E\left(\pi,q_2'\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots((6.13)$$

$$\therefore \quad I' = \pi\ln\frac{2G_1a+b^2}{2G_1a-b^2} +$$

$$\sqrt{H}\left\{\left(\frac{2G_1a+b^2}{4a}+\frac{4a}{2G_1a+b^2}\right)\sqrt{K+1}\ E(\pi,q_1) - \left(\frac{2G_1a-b^2}{4a}+\frac{4a}{2G_1a-b^2}\right)\sqrt{K_1+1}\ E(\pi,q_2)\right\}$$

and

$$I'' = \pi\ln\frac{2G_1a+b^2}{2G_1a-b^2} +$$

$$\sqrt{H}\left\{\left(\frac{2G_1a+b^2}{4a}+\frac{4a}{2G_1a+b^2}\right)\sqrt{K'+1}\,E\left(\left(\pi,q_1'\right)-\left(\frac{2G_1a-b^2}{4a}+\frac{4a}{2G_1a-b^2}\right)\sqrt{K_1'+1}\,E\left(\pi,q_2'\right)\right)\right\}$$

$$I = I' - I''$$

$$= -G\rho\left[\begin{array}{l}\left(\dfrac{2G_1a+b^2}{4a}+\dfrac{4a}{2G_1a+b^2}\right)\left\{\sqrt{H}\sqrt{K+1}\,E(\pi,q_1)-\sqrt{H}\sqrt{K'+1}\,E\left(\pi,q_1'\right)\right\}\\ -\left(\dfrac{2G_1a-b^2}{4a}+\dfrac{4a}{2G_1a-b^2}\right)\left\{\sqrt{H}\sqrt{K_1+1}\,E(\pi,q_2)-\sqrt{H}\sqrt{K_1'+1}\,E\left(\pi,q_2'\right)\right\}\end{array}\right]$$

....(6.14)

When, $E(\pi,q) = \pi\left\{1-\dfrac{1^2}{2^2}q^2-\dfrac{1^2.3}{2^2.4}q^4-\text{.........................}\right\}$

Now, $\sqrt{H}\sqrt{K+1} = \sqrt{\left(\dfrac{G_1}{2}+\dfrac{b^2}{4a}\right)^2+D+H}$

$$= \sqrt{\left(-RCos\phi'+\frac{b^2}{4a}\right)^2+R^2Sin^2\phi'+b^2-2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2+16a^2b^2+b^4-8ab^2RCos\varphi'-32a^2RSin\varphi'}$$

Now,

$$\sqrt{H}\sqrt{K'+1} = \sqrt{\left(-RCos\phi' + \frac{b^2}{4a}\right)^2 + R^2 Sin^2\phi' - 2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2 + b^4 - 8ab^2 RCos\varphi' - 32a^2 RSin\varphi'} \ ,$$

$$\sqrt{H}\sqrt{K_1+1} = \sqrt{\left(-RCos\phi' - \frac{b^2}{4a}\right)^2 + R^2 Sin^2\phi + b^2 - 2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2 + 16a^2b^2 + b^4 + 8ab^2 RCos\varphi' - 32a^2 RSin\varphi'}$$

and

$$\sqrt{H}\sqrt{K_1'+1} = \sqrt{\left(-RCos\phi' - \frac{b^2}{4a}\right)^2 + R^2 Sin^2\phi' - 2RSin\phi'}$$

$$= \frac{1}{4a}\sqrt{16a^2R^2 + b^4 + 8ab^2 RCos\varphi' - 32a^2 RSin\varphi'}$$

Hence, finally we get

$$\psi = -\frac{G\rho}{4a}\left[\left(\frac{-4aRCos\varphi' + b^2}{4a} + \frac{4a}{-4aRCos\varphi' + b^2}\right)\times\right.$$

$$\left\{\sqrt{16a^2R^2 + 16a^2b^2 + b^4 - 8ab^2RCos\varphi' - 32a^2RSin\varphi'}\ E(\pi, q_1)\right.$$

$$\left.-\sqrt{16a^2R^2 + b^4 - 8ab^2RCos\varphi' - 32a^2RSin\varphi'}\ E(\pi, q_1')\right\}$$

$$+\left(\frac{4aRCos\varphi' + b^2}{4a} + \frac{4a}{4aRCos\varphi' + b^2}\right)\times$$

$$\left\{\sqrt{16a^2R^2 + 16a^2b^2 + b^4 + 8ab^2RCos\varphi' - 32a^2RSin\varphi'}\ E(\pi, q_2)\right.$$

$$\left.\left.-\sqrt{16a^2R^2 + b^4 + 8ab^2RCos\varphi' - 32a^2RSin\varphi'}\ E(\pi, q_2')\right\}\right]$$

(6 15)

Where,

$$q_1^2 = \frac{-64a^2RSin\varphi'}{16a^2R^2 + 16a^2b^2 + b^4 - 8ab^2RCos\varphi' - 32a^2RSin\varphi'}$$

$$q_1'^2 = \frac{-64a^2 RSin\varphi'}{16a^2R^2 + b^4 - 8ab^2 RCos\varphi' - 32a^2 RSin\varphi'}$$

$$q_2^2 = \frac{-16a^2 RSin\varphi'}{16a^2R^2 + 16a^2b^2 + b^4 - 8ab^2 RCos\varphi' - 32a^2 RSin\varphi'}$$

$$q_2'^2 = \frac{-64a^2 RSin\varphi'}{16a^2R^2 + b^4 + 8ab^2 RCos\varphi' - 32a^2 RSin\varphi'}$$

Thus, equation (6.15) is the expression for Gravitational Potential of an elliptical mass of Oblate shape at any point outside the Oblate. In Chapter-3 (Sec. 3.3), in case of the calculation of Gravitational Potential of thin uniform straight rod at any point, we have seen that if we draw an ellipse with A and B as foci then for all points on the ellipse, the sum of the focal distances will be the same (where for the point P on the surface of the ellipse the sum of the focal distances is $(r_1 + r_2)$) and for all points over the surface of the ellipse the potential is constant. The potential is also constant over the ellipsoid obtained by revolving this ellipse about the line joining the foci, i.e., about AB.

In figure 6.2, ψ vs R is plotted; $\varphi =$ $30^0, 45^0$ and 90^0. Figure 6.3 shows the plot of ψ vs φ. These figures(using FORTRAN – LF9556 Compiler) show the variation of the Gravitational Potential with respect to angle φ and distance R.

In Figure 6.3, we see that the value of ψ drops to a minimum and then rises again. This may be because of the fact that the potential remains constant only in some ellipsoidal surface like in case of an uniform rod as discussed above, whereas in this case the distance R is kept constant and it will generate a spherical surface.

Thus we find that the expression for Gravitational Potential of an elliptical mass of Oblate shape at any point outside the Oblate (equation (6.15)) is correct.

The programme is given in Appendix-II & -III

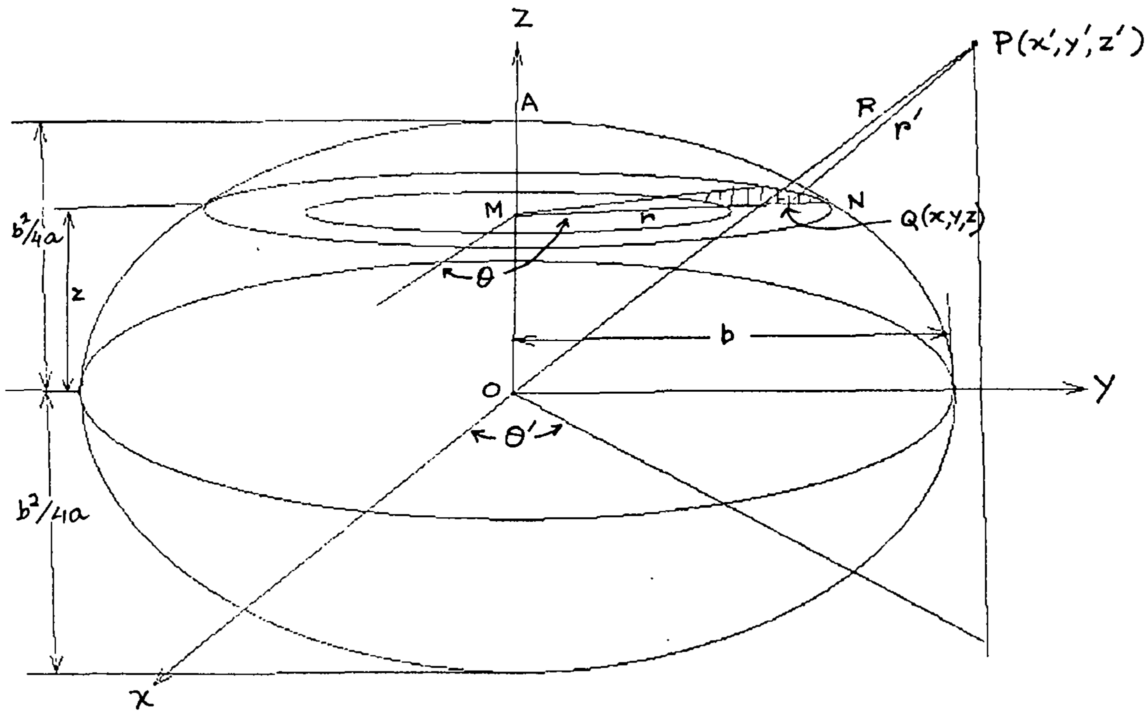

Figure 6 1   Oblate Shaped Mass

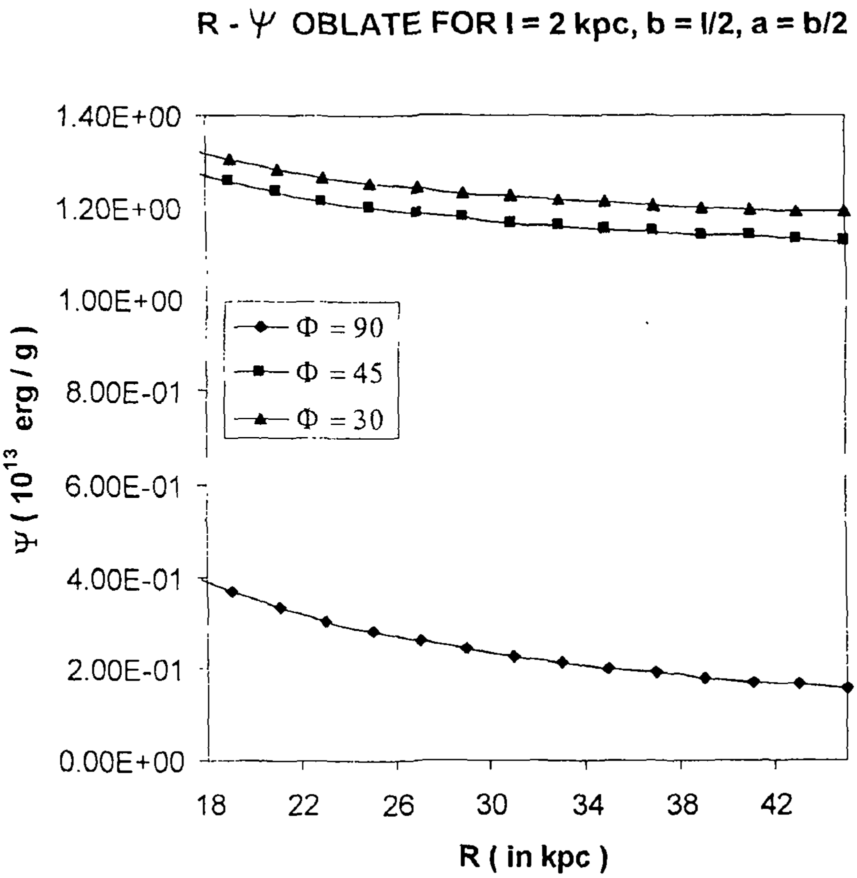

Figure 6.2 Plot of Ψ (10^{13} erg / g) and R (in kpc) for Oblate

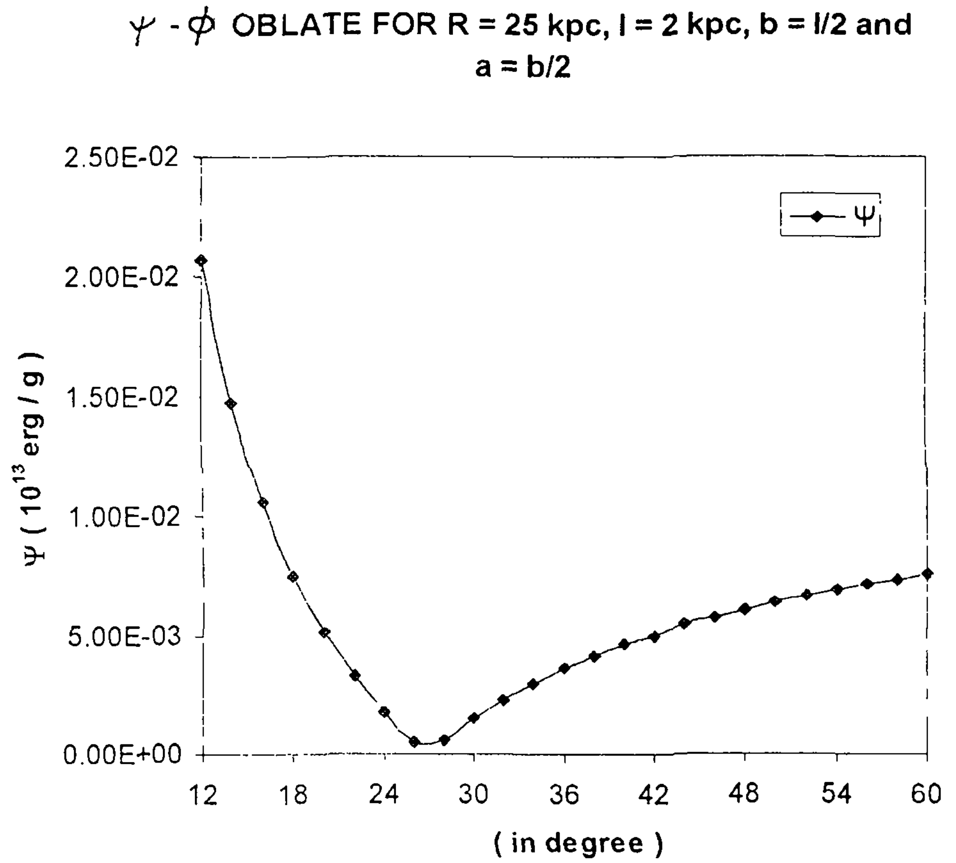

Figure 6.3 Plot of Ψ (10^{13} erg / g) and Φ (in degree) for Oblate

7

Summary and Conclusion

The Newton's law is one of the greatest discoveries which scientific investigation ever yielded to mankind. According to this law, every matter of particle attracts every other particle with a force which is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

The astronomical observations of the Danish astronomer Tycho Brahe on the motion of Mars led Kepler to formulate the law of planetary motion. Newton gave us an explanation of planetary motion in terms of the force of gravity which exists between the sun and the planet. Thus, Newton's law of gravitation hold even when r is large (except in the case, where relativistic corrections are appreciable). It should also be noted that Newton's law is valid even when r is very small but ceases to hold when r is of the order of 10^{-7} cm In this case, instead of attraction, there is repulsion, because if there is attraction between two atoms separated by a distance of the order of 10^{-7} cm. the atoms will superimpose on each other which is practically impossible. Actually the force in this case is of electrical nature.

Thus we see that the law is applicable to all distances ranging from zero to infinity

For example, the law of gravitation is true at even very large distances like the distance between galaxies. Gravity appears to exist at even distances of tens of millions of light years. In a galaxy we have a scale of 50,000 to 100,000 light years for measuring distances between heavenly bodies. The earth's distance from the sun is $8\frac{1}{2}$ light minutes, so we see how large these dimensions are. It should be noted that the law, unlike Inverse Square law; as in the case of electrostatics and magnetism, is not influenced by the medium in which the force acts. It is also independent of chemical composition of the attracting masses, and their temperatures. Thus we find that the law is universal.

In a gravitational field, to move a unit mass from one point to another point requires an expenditure of work This amount of work done may be negative or positive according as the body is moved in the direction or against the direction of the force of attraction. This amount of work done is a measure of the potential difference between the two points between which the unit mass is moved. The gravitational potential at a point in a gravitational field is measured by the amount of work done in bringing a unit mass from infinity to that point

All the calculation on Gravitational Potential at points outside non-spherical masses like disc or cylinder, available in existing literature, are done considering the points either on the axis or on the plane of the disc/cylinder However, it is required to be studied at any point outside such bodies

In this context in Chapter–1, we have briefly discussed the discovery of the laws of Gravitation and its consequences, its history and refinements of Gravitational laws by Einstein

In Chapter-2, we have briefly discussed about the Gravitational Potential, its physical meaning, equipotential surface, Laplace's and Poisson's equations for potential

In Chapter-3, we have discussed the methods available in literature and text books of finding Gravitational Potential at any point on the axis or on the plane of the non-spherical masses like disc and cylinder

In Chapter-4 we have calculated the gravitational potential of a solid cylinder at any point not required to be only on the plane or on axis, outside the body (Figure 4.1). In figure 4.2, ψ vs R is plotted. Figure 4.3 and 4.4 show the plot of ψ vs φ.These figures (using FORTRAN – LF9556 Compiler) show the variation of gravitational potential with respect to angle φ and distance R.

Similarly, in Chapter-5 , we have calculated the gravitational potential of an ellipsoidal mass of Prolate shape at any point outside the Prolate (Figure 5.1). In figure 5.2, ψ vs R is plotted. Figure 5.3 shows the plot of ψ vs φ. These figures (using FORTRAN – LF9556 Compiler) show the variation of gravitational potential with respect to angle φ and distance R.

In Chapter-6 , we have calculated the gravitational potential of an ellipsoidal mass of Oblate shape at any point outside the Oblate (Figure 6.1). In figure 6.2, ψ vs R is plotted. Figure 6.3 shows the plot of ψ vs φ.These figures (using FORTRAN – LF9556 Compiler) show the variation of gravitational potential with respect to angle φ and distance R.

Equations (4.14), (5.15) and (6.15) are the expressions for Gravitational Potential of a Cylindrical mass, an elliptical mass of Prolate shape and an elliptical mass of Oblate shape at any point outside the Cylinder, Prolate and Oblate respectively. In Chapter-3 (Sec. 3.3), in case of the calculation of Gravitational Potential of thin uniform straight rod at any point, we have seen that if we draw an ellipse with A and B as foci then for all points on the ellipse, the sum of the focal distances will be the same [where for the point

P on the surface of the ellipse the sum of the focal distances is $(r_1 + r_2)$] and for all points over the surface of the ellipse the potential is constant. The potential is also constant over the ellipsoid obtained by revolving this ellipse about the line joining the foci, i.e., about AB.

In Figure 5.3 and 6.3, we see that the value of ψ drops to a minimum and then rises again. This may be because of the fact that the potential remains constant only in some ellipsoidal surface like in case of an uniform rod as discussed in Chapter-3 (Sec. 3.3), whereas in this case the distance R is kept constant and it will generate a spherical surface.

Thus we find that the expressions for Gravitational Potential of an elliptical mass of Prolate shape and an elliptical mass of Oblate shape at any point outside the Prolate (equation (5.15) and Oblate (equation (6.15) are correct.

A knowledge of the Gravitational Potential due to such bodies will help in understanding the dynamics of cluster of galaxies and Gravitational bending of light due to galaxies of various shapes.

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APPENDIX-I: FORTRAN PROGRAMM WRITTEN BY USING LF9556 COMPILER

```
C       LAST CHANGE:  DAS  13 JAN 2008    0:11 AM
C       TO CALCULATE THE POTENTIAL
C       PROGRAM WRITTEN BY RINKU CHAKRABORTY
        COMMON  R,L,M0,G,R0,RHO,S,C,PI,PHI,PHI1,SHI
        COMMON  T1,T2,T3,T4,Q1,Q2,Q3,Q4,E1,E2,E3,E4,A1,A2
        PI=22./7.
C       PHI=-1.
        PHI=0.0
C       PHI=PI/4.
C       PHI=PI/2.
C       DO 5 I=1,15
        DO 5 I=1,30
C       PHI=PHI+5.
        PHI=PHI+0.2
C       PHI=PHI+1.
        PHI1=PHI
        S=SIN(PHI1)
        C=COS(PHI1)
        OPEN (UNIT=7,FILE= "SHI.OUT")
        KPC=1.
        R=25.*KPC
        L=2.*KPC
        R0=15.*KPC
        M0=1.9E33
        M=(1.4E11)*M0
C       V=PI*R**2.*L
        G=6.67*10.E8
        RHO=1.095*10.E21
C       RHO=M/V
        T1=(4.*R**2.+L**2.-8.*R*S-4.*R*L*C)
        T2=4.*R0**2+T1
        T3=(4.*R**2.+L**2.-8.*R*S+4.*R*L*C)
        T4=4.*R0**2+T3
        Q1=(-16.*R*S)/T2
        Q2=(-16.*R*S)/T1
        Q3=(-16.*R*S)/T4
        Q4=(-16.*R*S)/T3
        E1=PI*(1.-(1./4.)*Q1-(3./64.)*(Q1)**2)
        E2=PI*(1.-(1./4.)*Q2-(3./64.)*(Q2)**2)
        E3=PI*(1.-(1./4.)*Q3-(3./64.)*(Q3)**2)
        E4=PI*(1.-(1./4.)*Q4-(3./64.)*(Q4)**2)
        A1=((-2.*R*C+L)**2.+4.)/(-2.*R*C+L)
        A2=((2.*R*C+L)**2.+4.)/(2.*R*C+L)
        A1= A1*(SQRT(T2)*E1-SQRT(T1)*E2)
        A2= A2*(SQRT(T4)*E3-SQRT(T3)*E4)
        A3=A1+A2
        SHI=-.25*G*RHO*A3
C       SHI=ABS(SHI)
        WRITE (7,9) PHI,A1,A2,A3,RHO,SHI
C       WRITE (7,9) R,A1,A2,A3,RHO,SHI
    9   FORMAT (1X,F10.4,5X,5E20.3)
    5   CONTINUE
        STOP
        END
```

APPENDIX-II, FORTRAN PROGRAMM WRITTEN BY USING LF9556 COMPILER

```
C       LAST CHANGE:  DAS  12 JAN 2008    0:39 AM
C       CALCULATION OF POTENTIAL
C       PROGRAM WRITTEN BY RINKU CHAKRAVARTY
        COMMON  R,L,M0,G,R0,RHO,S,C,PI,PHI,PHI1,SHI
        COMMON  T1,T2,T3,T4,Q1,Q2,A1,A2,A3
        PI=22./7.
C       PHI=-1.
C       PHI=0.0
        PHI=PI/4.
C       PHI=PI/2.
C       R=0.0
        DO 5 I=1,15
C       R=R+1.
C       R=R+5.
C       R=R+5.
C       R=R+.1
C       DO 5 I=1,15
        PHI=PHI+5.
C       PHI=PHI+0.1
C       PHI=PHI+1.
        PHI1=PHI
        S=SIN(PHI1)
        C=COS(PHI1)
        OPEN (UNIT=7,FILE= "SHI.OUT")
        PC=3.085678*(10.**18.)
        KPC=1000.
        R=25.*KPC
        L=2.*KPC
C       M0=1.9*10**33.
C       M=(1.4*10**11)*M0
C       A=(L/2.)
C       B=(A/2.)
        B=(L/2.)
        A=(B/2.)
        G=6.67*10.**(-8.)
        V=PI*R**2.*L
        M0=1.9E33
        M=(1.4E11)*M0
        RHO=M/V
      T1=( SQRT (16.*A**2.*R**2.+16.*A**2.*B**2+B**4.-8.*A*B**2.*R*C
     $  -32.*A**2.*R*S))
        Q1=((-64.*A**2.*R*S)/(T1)**2.)
        E1=PI*(1.-(1./4.)*Q1-(3./64.)*(Q1)**2)
        T111=T1*E1
        T2=(1./4.*A)*T111
        T3=(((-4.*A*R*C+B**2.)/4.*A)+(4.*A/(-4.*A*R*C+B**2.)))*T2
        T4= (SQRT (16.*A**2.*R**2.+B**4.-8.*A*B**2.*R*C-32.*A**2.*R*S))
        Q2=((-64.*A**2.*R*S)/(T4)**2.)
        E2=PI*(1.-(1./4.)*Q2-(3./64.)*(Q2)**2)
        T222=T4*E2
        T5=(1./4.*A)*T222
        T6=(((-4.*A*R*C+B**2.)/4.*A)+(4.*A/(-4.*A*R*C+B**2.)))*T5
        T7=( SQRT (16.*A**2.*R**2.+16.*A**2.*B**2+B**4.+8.*A*B**2.*R*C
     $  -32.*A**2.*R*S))
        Q3=((-64.*A**2.*R*S)/(T7)**2.)
```

```
      E3=PI*(1.-(1./4.)*Q3-(3./64.)*(Q3)**2)
      T333=T7*E3
      T8=(1./4.*A)*T333
      T9=(((-4.*A*R*C-B**2.)/4.*A)+(4.*A/(-4.*A*R*C-B**2.)))*T8
      T10= (SQRT (16.*A**2.*R**2.+B**4.+8.*A*B**2.*R*C-32.*A**2.*R*S))
      Q4=((-64.*A**2.*R*S)/(T10)**2.)
      E4=PI*(1.-(1./4.)*Q2-(3./64.)*(Q4)**2)
      T444=T10*E4
      T11=(1./4.*A)*T444
      T12=(((-4.*A*R*C-B**2.)/4.*A)+(4.*A/(-4.*A*R*C-B**2.)))*T11
      SHI=((T3-T6)+(T9-T12))
C     SHI=-.25*G*RHO*A3
      SHI=ABS(SHI)
C     WRITE (7,9) R,A1,A2,A3,RHO,SHI
      WRITE (7,9) PHI,A1,A2,A3,RHO,SHI
    9 FORMAT (1X,F10.4,5X,5E20.4)
C   9 FORMAT (1X,F10.4,5X,11F10.5)
  5   CONTINUE
      STOP
      END
```

APPENDIX-III, FORTRAN PROGRAMME WRITTEN BY USING LF9556 COMPILER

```
C         LAST CHANGE:  DAS  12 JAN 2008    7:53 PM
C         CALCULATION OF POTENTIAL
C         PROGRAM WRITTEN BY RINKU CHAKRAVARTY
          COMMON  R,L,M0,G,R0,RHO,S,C,PI,PHI,PHI1,SHI
          COMMON  T1,T2,T3,T4,Q1,Q2,A1,A2,A3
          PI=22./7.
C         PHI=PI/4.
C         PHI=PI/3.
C         PHI=PI/2.
C         PHI=PI/6.
          PHI=0.0
C         KPC=(10**3.)
          KPC=1.
          L=3.*KPC
          AN=5.0
          R=AN*L
C         DO 5 I=1,25
          DO 5 I=1,15
          R=R+2.
C         R=R+5.
          PHI1=PHI
          S=SIN(PHI1)
          C=COS(PHI1)
          OPEN (UNIT=7,FILE= "SHI.OUT")
C         PC=3.085678*(10.**18.)
C         KPC=1000.
C         R=25.*KPC
C         L=2.*KPC
          M0=1.9*(10.**33.)
          M =1.4*(10.**11.)*M0
          G=6.67*(10.**(-8.))
C         A=(L/2.)
C         B=(A/2.)
          B=(L/2.)
          A=(B/2.)
          V=PI*R**2.*L
          RHO=1.095*(10.**21.)
         T1=( SQRT (16.*A**2.*R**2.+16.*A**2.*B**2+B**4.-8.*A*B**2.*R*C
     $    -32.*A**2.*R*S))
          Q1=((-64.*A**2.*R*S)/(T1)**2.)
          E1=PI*(1.-(1./4.)*Q1-(3./64.)*(Q1)**2)
          T111=T1*E1
          T2=(1./4.*A)*T111
          T3=(((-4.*A*R*C+B**2.)/4.*A)+(4.*A/(-4.*A*R*C+B**2.)))*T2
          T4= (SQRT (16.*A**2.*R**2.+B**4.-8.*A*B**2.*R*C-32.*A**2.*R*S))
          Q2=((-64.*A**2.*R*S)/(T4)**2.)
          E2=PI*(1.-(1./4.)*Q2-(3./64.)*(Q2)**2)
          T222=T4*E2
          T5=(1./4.*A)*T222
          T6=(((-4.*A*R*C+B**2.)/4.*A)+(4.*A/(-4.*A*R*C+B**2.)))*T5
          T7=( SQRT (16.*A**2.*R**2.+16.*A**2.*B**2+B**4.+8.*A*B**2.*R*C
     $    -32.*A**2.*R*S))
          Q3=((-64.*A**2.*R*S)/(T7)**2.)
          E3=PI*(1.-(1./4.)*Q3-(3./64.)*(Q3)**2)
          T333=T7*E3
```

```
      T8=(1./4.*A)*T333
C     T9=(((-4.*A*R*C-B**2.)/4.*A)+(4.*A/(-4.*A*R*C-B**2.)))*T8
      T9=(((4.*A*R*C+B**2.)/4.*A)+(4.*A/(4.*A*R*C+B**2.)))*T8
      T10= (SQRT (16.*A**2.*R**2.+B**4.+8.*A*B**2.*R*C-32.*A**2.*R*S))
      Q4=((-64.*A**2.*R*S)/(T10)**2.)
      E4=PI*(1.-(1./4.)*Q2-(3./64.)*(Q4)**2)
      T444=T10*E4
      T11=(1./4.*A)*T444
C     T12=(((-4.*A*R*C-B**2.)/4.*A)+(4.*A/(-4.*A*R*C-B**2.)))*T11
      T12=(((4.*A*R*C+B**2.)/4.*A)+(4.*A/(4.*A*R*C+B**2.)))*T11
      SHI=((T3-T6)+(T9-T12))
C     SHI=-.25*G*RHO*A3
      SHI=ABS(SHI)
      WRITE (7,9) R,T3,T6,T9,T12,SHI
C     WRITE (7,9) PHI,T3,T6,T9,T12,SHI
    9 FORMAT (1X,F10.4,5X,5E20.4)
C   9 FORMAT (1X,F10.4,5X,11F10.5)
  5   CONTINUE
      STOP
      END
```

TH-848
19/12/12